A model of optimal consumption under liquidity risk with random trading times and its coupled system of integrodifferential equations

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Abstract

We consider a portfolio/consumption choice problem in a market model with liquidity risk. The main feature is that the investor can trade and observe stock prices only at exogenous Poisson arrival times. He may also consume continuously from his cash holdings, and his goal is to maximize his expected utility from consumption. This is a mixed discrete/continuous stochastic control problem, nonstandard in the literature. We show how the dynamic programming principle leads to a coupled system of Integro-Differential Equations (IDE), and we prove an analytic characterization of this control problem by adapting the concept of viscosity solutions. We also provide a convergent numerical algorithm for the resolution to this coupled system of IDE, and illustrate our results with some numerical experiments.

Key words : liquidity, portfolio/consumption problem, integrodifferential equations, viscosity solutions.

MSC Classification (2000) : 93E20, 49K22, 49L25, 91B28.

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1 Introduction

A fundamental assumption of the traditional portfolio/consumption choice paradigm of Merton [11] is that assets are liquid and readily continuously tradable by economic agents. In reality, there are some restrictions on securities trade, and investors cannot buy and sell them immediately; typical examples of assets in which trading is problematic include human capital, mutual funds, pension plans, inheritances, and residential real-estate. We then usually speak about liquidity risk meaning that one may have to wait some time before being able to unwind a position in some financial assets.

There are various approaches to model liquidity risk since it is in fact related to many factors. A familiar approach in the academic literature is to measure illiquidity in terms of bid-ask spread and transaction costs, see Davis and Norman [5], Jouini and Kallal [8] and many others. In this setting, potentially high cost is associated to frequent trading but the investors can trade whenever desired. On the other hand, there are some studies where illiquidity is represented by restrictions on trade times. For instance, Schwartz and Tebaldi [13] and Longstaff [9] assume in their model that illiquid assets can only be traded at the starting date and at a fixed terminal horizon. In a less extreme modelling, Rogers and Zane [12] and Matsumoto [10] consider random trade times by assuming that trade succeeds only at the jump times of a Poisson process, and study the impact on a portfolio choice problem. In these models, the price process is observed continuously, trading strategies are in continuous-time, and the corresponding portfolio/consumption problem leads to a standard jump-diffusion control problem, see also Wang [14]. However, illiquidity is often viewed by practitioners as the situation where their ability to trade assets is limited or restricted to the times when a quote comes into the market.

In this paper, we consider a description of liquidity risk which is consistent with the market-microstructure oriented modelling of high frequency financial data such as tick-by-tick stock prices. We assume that stock prices can be observed and traded only at random times of a Poisson process corresponding to quotes in the market. This setup is inspired by recent papers of Frey and Runggaldier [7] and Cvitanic, Liptser and Rozovskii [4], who assume in addition that there is an unobservable stochastic volatility, and are interested in the estimation of this volatility. In our liquidity risk context, we suppose that the investor is also allowed to consume continuously from the bank account, and we study the Merton's problem of maximizing the expected discounted utility of consumption.

From a mathematical viewpoint, the resulting optimization problem is a mixed discrete/continuous stochastic control problem, nonstandard in the literature. We show how it leads, via a dynamic programming principle, to a coupled system of nonlinear integro-partial differential equations (IPDE), for which we prove a classical verification theorem. Then, following the modern approach of stochastic control, and to overcome the possible lack of regularity of the value functions, we adapt the notion of viscosity solutions to our context, and prove a characterization (with a new uniqueness result) of the value functions to their coupled system of IPDE. We also provide a convergent numerical algorithm for solving this coupled system, and illustrate our results with some numerical experiments. In particular, we compare the value function and the optimal investment policy obtained in presence of liquidity risk with the ones in the classical Merton's model.

The plan of the paper is as follows. We formulate the liquidity risk model and the portfolio/consumption problem in Section 2. We show in Section 3 how it leads to a coupled system of IPDE and state the corresponding verification theorem. In Section 4, we provide an analytic characterization of the value function by means of viscosity solutions. Section

5 describes a convergent numerical algorithm and we give some numerical illustrations in Section 6.

2 Model and problem formulation

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ satisfying the usual conditions. All stochastic processes involved in this paper are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

We consider a model of an illiquid market where the investor can observe the positive stock price process S and trade only at random times $\{\tau_k\}_{k\geq 0}$ with $\tau_0 = 0 < \tau_1 < \ldots < \tau_k < \ldots$. For simplicity, we assume that S_0 is known, and we denote

$$Z_k \quad = \quad \frac{S_{\tau_k} - S_{\tau_{k-1}}}{S_{\tau_{k-1}}}, \quad k \geq 1,$$

the observed return process valued in $(-1, \infty)$, where we set by convention Z_0 to some fixed constant.

The investor may also consume continuously from the bank account (interest rate is assumed w.l.o.g to be zero) between two trading dates. We introduce the continuous observation filtration $\mathbb{G}^c = (\mathcal{G}_t)_{t\geq 0}$ with :

$$\mathcal{G}_t \quad = \quad \sigma\left\{(\tau_k, Z_k) \; : \tau_k \leq t\right\},$$

and the discrete observation filtration $\mathbb{G}^d = (\mathcal{G}_{\tau_k})_{k\geq 0}$. Notice that $\mathcal{G}_t$ is trivial for $t < \tau_1$.

A control policy is a mixed discrete-continuous process (α, c), where $\alpha = (\alpha_k)_{k\geq 1}$ is real-valued $\mathbb{G}^d$-predictable, i.e. α_k is $\mathcal{G}_{\tau_{k-1}}$-measurable, and $c = (c_t)_{t\geq 0}$ is a nonnegative $\mathbb{G}^c$-predictable process : α_k represents the amount of stock invested for the period $(\tau_{k-1}, \tau_k]$ after observing the stock price at time τ_{k-1}, and c_t is the consumption rate at time t based on the available information. Starting from an initial capital $x \geq 0$, and given a control policy (α, c), we denote X_k^x the wealth of the investor at time τ_k defined by :

$$X_k^x \quad = \quad x - \int_0^{\tau_k} c_t dt + \sum_{i=1}^{k} \alpha_i Z_i, \quad k \geq 1, \quad X_0^x \; = \; x. \tag{2.1}$$

Given $x \geq 0$, we say that a control policy (α, c) is admissible, and we denote $(\alpha, c) \in \mathcal{A}(x)$ if :

$$X_k^x \quad \geq \quad 0, \quad a.s. \quad \forall\, k \geq 1. \tag{2.2}$$

We are interested in the optimal portfolio/consumption problem :

$$v(x) \quad = \quad \sup_{(\alpha,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right], \quad x \; \geq 0, \tag{2.3}$$

where $\rho > 0$ is a positive discount factor, and U is an utility function defined on $\mathbb{R}_+$, with w.l.o.g. $U(0) = 0$, nondecreasing, concave and C^1 on $(0, \infty)$ satisfying the Inada conditions $U'(0^+) = \infty$ and $U'(\infty) = 0$. We shall assume the following growth condition on U : there exists $\gamma \in (0, 1)$ s.t.

$$U(x) \quad \leq \quad K_1 x^\gamma, \quad x \geq 0, \tag{2.4}$$

for some positive constant K_1. We denote $I = (U')^{-1} : (0,\infty) \mapsto (0,\infty)$ the inverse function of the derivative U', and $\tilde{U}$ the convex conjugate of U i.e. :

$$\tilde{U}(y) \quad = \quad \sup_{x>0}[U(x) - xy] \quad = \quad \begin{cases} U(I(y)) - yI(y), & y > 0 \\ U(\infty), & y = 0 \\ \infty, & y < 0 \end{cases} \tag{2.5}$$

Notice that $\tilde{U}$ is nonincreasing, $\tilde{U}(\infty) = U(0)$, and under (2.4) we have

$$\tilde{U}(y) \quad \leq \quad \tilde{K}_1 y^{-\tilde{\gamma}}, \quad y \geq 0, \quad \text{with } \tilde{\gamma} \; = \; \frac{\gamma}{1-\gamma} \; > \; 0, \tag{2.6}$$

for some positive constant $\tilde{K}_1$ (actually $\tilde{K}_1 = \frac{(K_1\gamma)^{\frac{1}{1-\gamma}}}{\tilde{\gamma}}$).

Problem (2.3) is a mixed discrete/continuous-time stochastic control problem : this is a nonstandard control problem, which was not yet studied in the literature (to the best of our knowledge). In particular, we cannot derive as usual the Backward or Bellman equation associated to (2.3). Our paper is a first attempt to study such a control problem. In the rest of the paper, the following conditions on (τ_k, Z_k) stand in force.

(H1) $\{\tau_k\}_{k\geq 1}$ is the sequence of jump times of a Poisson process with intensity λ.

(H2) **(i)** For all $k \geq 1$, conditionally on the interarrival time $\tau_k - \tau_{k-1} = t \in \mathbb{R}_+$, Z_k is independent from $\{\tau_i, Z_i\}_{i<k}$ and has a distribution denoted $p(t,dz)$.
(ii) For all $t \geq 0$, the support of $p(t,dz)$ is
- either an interval with interior equal to $(-\underline{z}, \bar{z})$, $\underline{z} \in (0,1]$ and $\bar{z} \in (0,\infty]$,
- or is finite equal to $\{-\underline{z}, \ldots, \bar{z}\}$, $\underline{z} \in (0,1]$ and $\bar{z} \in (0,\infty)$.

(H3) $\int zp(t,dz) \geq 0$, for all $t \geq 0$, and there exist some $\kappa \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$ s.t.

$$\int (1+z)p(t,dz) \leq \kappa e^{bt}, \quad \forall t \geq 0.$$

The last condition **(H3)** means that for all $k \geq 1$,

$$1 \quad \leq \quad \mathbb{E}\Big[\frac{S_{\tau_k}}{S_{\tau_{k-1}}}\Big|\tau_k - \tau_{k-1} = t\Big] \quad \leq \quad \kappa e^{bt}, \quad \forall t \geq 0.$$

Example 2.1. S is extracted from a Black-Scholes model : $dS_t = bS_t dt + \sigma S_t dW_t$, with $b \geq 0$, $\sigma > 0$. Then $p(t,dz)$ is the distribution of

$$Z(t) \quad = \quad \exp\left[\left(b - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] - 1,$$

with support $(-1,\infty)$, and **(H3)** is clearly satisfied, since in this case $\int(1+z)p(t,dz) = \mathbb{E}\big[\exp\big((b - \sigma^2/2)t + \sigma W_t\big)\big] = e^{bt}$.

Example 2.2. Z_k is independent of the waiting times $\tau_k - \tau_{k-1}$, in which case its distribution $p(dz)$ does not depend on t. In particular, $p(dz)$ may be a discrete distribution with support $\{z_0, \ldots, z_d\}$ s.t. $\underline{z} = -z_0 \in (0,1]$ and $z_d = \bar{z} \in (0,\infty)$.

Remark 2.1. 1) It is easy to see that if the support of Z_k is included in $(0, \infty)$, i.e. the sequence $(S_k)_k$ is increasing, or is included in $(-1, 0)$, i.e. $(S_k)_k$ is decreasing, then the value function v is infinite. Indeed, suppose that $\underline{z} > 0$. Then, one can consume as much as wanted, by buying enough actions in order to satisfy the admissibility condition, so that v is infinite. A similar argument is valid (by selling actions) when $\bar{z} < 0$.
2) The condition $\int zp(t, dz) \geq 0$ is simply put for financial interpretation, but could be relaxed (see Remarks 3.3 and 3.4). The other condition in **(H3)** is more crucial.

Remark 2.2. Since $X^x_{k+1} = X^x_k - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1} Z_{k+1}$, and by the condition **(H2)** on the support of Z_{k+1}, we see that the admissibility condition (2.2) is written as :

$$X^x_k - \int_{\tau_k}^{s} c_u du + \alpha_{k+1} z \quad \geq \quad 0, \quad \forall k \geq 0, \, \forall s \geq \tau_k, \, \forall z \in \{-\underline{z}, \bar{z}\}.$$

almost surely. This may be also formulated directly in terms of $(\alpha, c) \in \mathcal{A}(x)$ as :

$$-\frac{X^x_k}{\bar{z}} \;\leq\; \alpha_{k+1} \quad \leq \quad \frac{X^x_k}{\underline{z}}, \quad \forall k \geq 0, \tag{2.7}$$

$$\int_{\tau_k}^{s} c_u du \quad \leq \quad X^x_k - \ell(\alpha_{k+1}), \quad \forall k \geq 0, \, \forall s \geq \tau_k, \tag{2.8}$$

where we set for all $a \in \mathbb{R}$:

$$\ell(a) \quad = \quad \max(a\underline{z}, -a\bar{z}),$$

with the convention that $\max(a\underline{z}, -a\bar{z}) = a\underline{z}$ when $\bar{z} = \infty$. In particular, we see that for $x = 0$, $\mathcal{A}(0) = \{0, 0\}$ and so $v(0) = 0$.

3 A first-order coupled system of nonlinear IPDE

In this section, we derive the coupled system of Integro Partial Differential Equation (IPDE) that will be satisfied by the value function of our control problem. The starting point is the following version of the dynamic programming principle (DPP) adapted to our context :

$$v(x) \quad = \quad \sup_{(\alpha,c) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(X^x_1)\right]. \tag{3.1}$$

This DPP will be proved rigorously in Appendix. In this section, we derive formally the coupled integrodifferential system arising from the DPP, and state a verification theorem, which shows that a suitable solution to this system should be the value function of our orginal control problem. The converse property, proved in the next section, states that the original value function is characterized as the unique (viscosity) solution to the coupled integrodifferential system.

Now, from the expression (2.1) of the wealth, and the measurability conditions on the control, the above dynamic programming relation is written as

$$v(x) \quad = \quad \sup_{(a,c) \in A_d(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(x - \int_0^{\tau_1} c_t dt + a Z_1)\right], \tag{3.2}$$

where $A_d(x)$ is the set of pairs (a, c) with a deterministic constant, and c a deterministic nonnegative process s.t. (see Remark 2.2) $a \in [-x/\bar{z}, x/\underline{z}]$ and

$$\int_0^t c_u du \;\; \leq \;\; x - \ell(a) \quad \text{i.e.} \quad x - \int_0^t c_u du + az \;\; \geq \;\; 0, \; \forall t \geq 0, \; \forall z \in (-\underline{z}, \bar{z}). \tag{3.3}$$

Given $a \in [-x/\bar{z}, x/\underline{z}]$, we denote by $\mathcal{C}_a(x)$ the set of deterministic nonnegative processes satisfying (3.3). Moreover, under conditions **(H1)** and **(H2)**, we may explicit (see also details in Lemma 3.1) the r.h.s. of (3.2) so that :

$$v(x) \;\; = \;\; \sup_{\substack{a \in \left[-\frac{x}{\bar{z}}, \frac{x}{\underline{z}}\right] \\ c \in \mathcal{C}_a(x)}} \int_0^\infty e^{-(\rho+\lambda)t} \left[U(c_t) + \lambda \int v(x - \int_0^t c_s ds + az) p(t, dz) \right] dt. \tag{3.4}$$

Let

$$\mathcal{D} \;\; = \;\; \mathbb{R}_+ \times \mathcal{X} \quad \text{with} \;\; \mathcal{X} \;\; = \;\; \left\{ (x, a) \in \mathbb{R}_+ \times \mathbb{R} \;\; : -\frac{x}{\bar{z}} \leq a \leq \frac{x}{\underline{z}} \right\}. \tag{3.5}$$

By setting $A = \mathbb{R}$ if $\bar{z} < \infty$, and $A = \mathbb{R}_+$ if $\bar{z} = \infty$, notice that $\mathcal{X}$ is written also as

$$\mathcal{X} \;\; = \;\; \{(x, a) \in \mathbb{R}_+ \times A \;\; : x \geq \ell(a)\}.$$

Now, we introduce the dynamic auxiliary control problem : for $(t, x, a) \in \mathcal{D}$,

$$\hat{v}(t, x, a) \;\; = \;\; \sup_{c \in \mathcal{C}_a(t,x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int v(Y_s^{t,x} + az) p(s, dz) \right] ds, \tag{3.6}$$

where $\mathcal{C}_a(t, x)$ is the set of deterministic nonnegative processes $c = (c_s)_{s \geq t}$ s.t.

$$\int_t^s c_u du \;\; \leq \;\; x - \ell(a) \quad \text{i.e.} \;\; Y_s^{t,x} + az \;\; \geq \;\; 0, \quad \forall s \geq t, \; \forall \, z \in (-\underline{z}, \bar{z}), \tag{3.7}$$

and $Y^{t,x}$ is the deterministic controlled process by $c \in \mathcal{C}_a(t, x)$:

$$Y_s^{t,x} \;\; = \;\; x - \int_t^s c_u du, \quad s \geq t. \tag{3.8}$$

We shall see later (see Lemma 4.2) that $\hat{v}$ lies in $C_+(\mathcal{D})$, the set of nonnegative continuous functions on $\mathcal{D}$. From (3.4)-(3.6), the original value function is then related to this auxiliary optimization problem by :

$$v \;\; = \;\; \mathcal{H}\hat{v} \tag{3.9}$$

where $\mathcal{H}$ is the operator mapping $C_+(\mathcal{D})$ into the set $\mathcal{B}_+(\mathbb{R}_+)$ of nonnegative measurable functions on $\mathbb{R}_+$ by :

$$\mathcal{H}\hat{w}(x) \;\; = \;\; \sup_{a \in [-x/\bar{z}, x/\underline{z}]} \hat{w}(0, x, a). \tag{3.10}$$

Actually, we shall see in Lemma 4.1 that v is continuous on $\mathbb{R}_+$, and so lies in $C_+(\mathbb{R}_+)$ the set of nonnegative and continuous functions on $\mathbb{R}_+$.

Remark 3.1. For a given $a \in A$, $\hat{v}$ is the value function of an optimal consumption/problem over an infinite horizon in a certain environment :

$$\hat{v}(t,x,a) \quad = \quad \sup_{c \in \mathcal{C}_a(t,x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)} V_a(s, Y_s^{t,x}, c_s) ds,$$

where V_a is a modified utility function depending not only on the current consumption rate c_s, but also on the cumulated consumption $\int c_s ds$.

At this stage, we may study the deterministic control problem (3.6) by standard dynamic programming methods : the associated Hamilton-Jacobi equation is

$$\sup_{c \geq 0} \left[-(\rho+\lambda)\hat{v} + \frac{\partial \hat{v}}{\partial t} - c\frac{\partial \hat{v}}{\partial x} + U(c) + \lambda \int v(x+az)p(t,dz) \right] \quad = \quad 0, \quad (t,x,a) \in \mathcal{D},$$

that may be rewritten as a first order Integro Partial Differential Equation (IPDE)

$$(\rho+\lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x+az)p(t,dz) \quad = \quad 0, \quad (t,x,a) \in \mathcal{D}. \qquad (3.11)$$

Remark 3.2. In the particular case where the distribution $p(t,dz) = p(dz)$ does not depend on t, then the above IPDE reduces to the integro ordinary differential equation for $\hat{v}(x,a)$:

$$(\rho+\lambda)\hat{v} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x+az)p(dz) \quad = \quad 0, \quad (t,x,a) \in \mathcal{D},$$

with $v(x) = \sup_{a \in [-x/\bar{z}, x/\underline{z}]} \hat{v}(x,a)$

We have then splitted our original stochastic optimization problem into two coupled tractable deterministic optimization problems : Problem (3.6) is a family over $a \in A$ of standard deterministic control problems on infinite horizon, which is stationary (i.e. $\hat{v}$ does not depend on t), whenever the distribution $p(t,dz)$ does not depend on t, and problem (3.9) is a classical one-dimensional extremum problem over a. Notice that these two optimization problems are coupled since the reward function appearing in the definition of problem (3.6) or in its IPDE (3.11) depends on the value function of problem (3.9) and vice-versa. However, this suggests a fixed point algorithm for solving our original optimization problem : this argument will be developed later.

The main result of this section is to validate the consistency of the above approach, by stating a verification theorem on the coupled IPDE (3.9)-(3.11). We proceed in several steps. We first state the following two lemmas.

Lemma 3.1. Assume **(H1)-(H2)** hold. Let $w \in \mathcal{B}_+(\mathbb{R}_+)$. Then, for any $x \geq 0$, $(\alpha, c) \in \mathcal{A}(x)$, $k \geq 0$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t) dt + e^{-\rho(\tau_{k+1}-\tau_k)} w(X_{k+1}^x) \middle| \mathcal{G}_{\tau_k} \right] \\
= \; &\int_{\tau_k}^\infty e^{-(\rho+\lambda)(t-\tau_k)} \left[U(c_t) + \lambda \int w\left(X_k^x - \int_{\tau_k}^t c_u du + \alpha_{k+1} z \right) p(t-\tau_k, dz) \right] dt.
\end{aligned}$$

Proof. Since $X^x_{k+1} = X^x_k - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1} Z_{k+1}$, we have by the law of conditional toy expectations :

$$
\begin{aligned}
& \mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t)dt + e^{-\rho(\tau_{k+1}-\tau_k)} w(X^x_{k+1}) \Big| \mathcal{G}_{\tau_k}\right] \\
= \ & \mathbb{E}\Bigg[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t)dt + \\
& \quad e^{-\rho(\tau_{k+1}-\tau_k)} \mathbb{E}\left[w\left(X^x_k - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1} Z_{k+1}\right) \Big| \mathcal{G}_{\tau_k}, \tau_{k+1}-\tau_k\right] \Big| \mathcal{G}_{\tau_k}\Bigg] \\
= \ & \mathbb{E}\Bigg[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t)dt + \\
& \quad e^{-\rho(\tau_{k+1}-\tau_k)} \int w\left(X^x_k - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1} z\right) p(\tau_{k+1}-\tau_k, dz) \Big| \mathcal{G}_{\tau_k}\Bigg] \\
= \ & \int_0^\infty \Bigg[\int_{\tau_k}^{\tau_k+s} e^{-\rho(t-\tau_k)} U(c_t)dt + \\
& \quad e^{-\rho s} \int w\left(X^x_k - \int_{\tau_k}^{\tau_k+s} c_u du + \alpha_{k+1} z\right) p(s, dz)\Bigg] \lambda e^{-\lambda s} ds,
\end{aligned}
$$

where we used **(H2)** in the second equality and **(H1)** in the last one. We conclude with Fubini's theorem and the change of variable $s \to s + \tau_k$. □

Lemma 3.2. Under **(H1)-(H2)-(H3)**, and (2.4), suppose that ρ satisfies

$$\rho \ > \ b\gamma + \lambda\big(\frac{\kappa^\gamma}{\underline{z}^\gamma} - 1\big). \tag{3.12}$$

Then, for all $x \geq 0$, $(\alpha, c) \in \mathcal{A}(x)$, we have

$$\mathbb{E}\left[e^{-\rho\tau_n}(X^x_n)^\gamma\right] \ \leq \ x^\gamma \delta^n, \tag{3.13}$$

where

$$\delta \ = \ \frac{\lambda}{\rho - b\gamma + \lambda} \frac{\kappa^\gamma}{\underline{z}^\gamma} \ < \ 1. \tag{3.14}$$

In particular, $\mathbb{E}\big[e^{-\rho\tau_n}(X^x_n)^\gamma\big]$ converges to 0, as n goes to ∞.

Proof. Observe from Jensen's inequality and conditions **(H2)-(H3)** that for all $x \geq 0$, $(\alpha, c) \in \mathcal{A}(x)$, $n \geq 1$,

$$
\begin{aligned}
\mathbb{E}\big[\big(X^x_{n-1} + \alpha_n Z_n\big)^\gamma \big| \mathcal{G}_{\tau_{n-1}}, \tau_n - \tau_{n-1}\big] \ & \leq \ \big(X^x_{n-1} + \alpha_n \int z \, p(\tau_n - \tau_{n-1}, dz)\big)^\gamma \\
& \leq \ \big(X^x_{n-1} + \frac{X^x_{n-1}}{\underline{z}}(\kappa e^{b(\tau_n - \tau_{n-1})} - 1)\big)^\gamma \\
& \leq \ (X^x_{n-1})^\gamma \frac{\kappa^\gamma}{\underline{z}^\gamma} e^{b\gamma(\tau_n - \tau_{n-1})}, \quad a.s.
\end{aligned} \tag{3.15}
$$

where we used also in the second inequality the bound (2.7) on α_n, and in the last one the fact that $\underline{z} \leq 1$. Thus, by writing that $X^x_n \leq X^x_{n-1} + \alpha_n Z_n$, and by the law of iterated conditional expectations, we get :

$$
\begin{aligned}
\mathbb{E}\big[e^{-\rho\tau_k}(X_n^x)^\gamma\big] &\le \mathbb{E}\big[e^{-(\rho-b\gamma)(\tau_n-\tau_{n-1})}e^{-\rho\tau_{n-1}}\frac{(X_{n-1}^x)^\gamma}{\underline{z}^\gamma}\kappa^\gamma\big] \\
&= \mathbb{E}\big[e^{-\rho\tau_{n-1}}\frac{(X_{n-1}^x)^\gamma}{\underline{z}^\gamma}\kappa^\gamma \int_0^\infty \lambda e^{-(\rho-b\gamma+\lambda)t}dt\big] \\
&= \delta\, \mathbb{E}\big[e^{-\rho\tau_{n-1}}(X_{n-1}^x)^\gamma\big]
\end{aligned}
$$

where we used condition **(H1)** in the first equality. We obtain the required result by induction on n, and the convergence since $\delta < 1$ under (3.12). $\Box$

Remark 3.3. In the case where $\int zp(t,dz) \le 0$, and by assuming $-\int zp(t,dz) \le \kappa e^{bt}$ for some κ, $b \in \mathbb{R}_+$, the inequality (3.15) should be replaced by :

$$
\mathbb{E}\big[\big(X_{n-1}^x + \alpha_n Z_n\big)^\gamma\big|\mathcal{G}_{\tau_{n-1}}, \tau_n - \tau_{n-1}\big] \le (X_{n-1}^x)^\gamma\big(1 + \frac{\kappa^\gamma}{\bar{z}^\gamma}e^{b\gamma(\tau_n-\tau_{n-1})}\big), \quad a.s.
$$

Then, by same arguments as in the above lemma, we obtain $\mathbb{E}\big[e^{-\rho\tau_k}(X_n^x)^\gamma\big] \le x^\gamma\delta^n$ with

$$
\delta = \frac{\lambda}{\rho+\lambda} + \frac{\lambda}{\rho - b\gamma + \lambda}\frac{\kappa^\gamma}{\bar{z}^\gamma}.
$$

Therefore, in this case, we get the convergence of $\mathbb{E}\big[e^{-\rho\tau_k}(X_n^x)^\gamma\big]$ to zero provided that

$$
\rho > b\gamma + \lambda\frac{\kappa^\gamma}{\bar{z}^\gamma}. \tag{3.16}
$$

The next result is a comparison principle for smooth solutions to the coupled IPDE (3.9)-(3.11).

Proposition 3.1. Under **(H1)-(H2)-(H3)**, (2.4) and (3.12), suppose there exists $\hat{w} \in C_+(\mathcal{D})$, C^1 with respect to (t,x), and $w \in C_+(\mathbb{R}_+)$ satisfying :

$$
(\rho+\lambda)\hat{w} - \frac{\partial\hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial\hat{w}}{\partial x}\right) - \lambda\int w(x+az)p(t,dz) \ge 0, \quad (t,x,a) \in \mathcal{D}, \tag{3.17}
$$

$$
w \ge \mathcal{H}\hat{w}, \tag{3.18}
$$

together with the growth condition :

$$
w(x) \le K(1+x^\gamma), \quad \forall x \ge 0, \tag{3.19}
$$

for some positive constant K. Then

$$
\hat{v} \le \hat{w} \quad \text{and} \quad v \le w.
$$

Proof. 1) Given $x \in \mathbb{R}_+$, for all $(\alpha, c) \in \mathcal{A}(x)$, apply, for any $k \ge 0$, standard differential calculus to $e^{-(\rho+\lambda)(s-\tau_k)}\hat{w}(s-\tau_k, Y_s^{(k)}, \alpha_{k+1})$ between τ_k and T (to be sent to infinity) where $Y_s^{(k)} = X_k^x - \int_{\tau_k}^s c_u du$:

$$
\begin{aligned}
& e^{-(\rho+\lambda)(T-\tau_k)}\hat{w}(T-\tau_k, Y_T^{(k)}, \alpha_{k+1}) \\
= \ & \hat{w}(0, X_k^x, \alpha_{k+1}) + \int_{\tau_k}^T e^{-(\rho+\lambda)(s-\tau_k)}\left[-(\rho+\lambda)\hat{w} + \frac{\partial\hat{w}}{\partial t} - c_s\frac{\partial\hat{w}}{\partial x}\right](s-\tau_k, Y_s^{(k)}, \alpha_{k+1})ds \\
\le \ & \hat{w}(0, X_k^x, \alpha_{k+1}) - \int_{\tau_k}^T e^{-(\rho+\lambda)(s-\tau_k)}\left[U(c_s) + \lambda\int w(Y_s^{(k)} + \alpha_{k+1}z)p(s-\tau_k, dz)\right]ds,
\end{aligned}
$$

from (3.17). Now, since $\hat{w}$ is nonnegative, we get by sending T to infinity :

$$\int_{\tau_k}^{\infty} e^{-(\rho+\lambda)(s-\tau_k)} \left[U(c_s) + \lambda \int w(Y_s^{(k)} + \alpha_{k+1} z) p(s - \tau_k, dz) \right] ds \quad \leq \quad \hat{w}(0, X_k^x, \alpha_{k+1}).$$

From Lemma 3.1, this is written as :

$$\begin{aligned} \mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(s-\tau_k)} U(c_s) ds + e^{-\rho(\tau_{k+1}-\tau_k)} w(X_{k+1}^x) \middle| \mathcal{G}_{\tau_k} \right] &\leq \hat{w}(0, X_k^x, \alpha_{k+1}) \\ &\leq w(X_k^x), \end{aligned}$$

where we used in the last inequality, (2.7) and the fact that $\mathcal{H}\hat{w} \leq w$. By induction on k and the law of iterated conditional expectations, we deduce

$$\mathbb{E}\left[\int_0^{\tau_n} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_n} w(X_n^x) \right] \quad \leq \quad w(x),$$

for all n. Now, from the growth condition (3.19) and Lemma 3.2, we have

$$\mathbb{E}\left[e^{-\rho\tau_n} w(X_n^x) \right] \quad \longrightarrow \quad 0, \tag{3.20}$$

as n goes to infinity. Therefore, we obtain

$$\mathbb{E}\left[\int_0^{\infty} e^{-\rho t} U(c_t) dt \right] \quad \leq \quad w(x),$$

which proves from the arbitrariness of (α, c) that $w \geq v$.
2) Given $(t, x, a) \in \mathcal{D}$, apply standard differential calculus to $e^{-(\rho+\lambda)(s-t)} \hat{w}(s, Y_s^{t,x}, a)$ between t and T (to be sent to infinity) where $Y_s^{t,x} = x - \int_t^s c_u du$, and c is arbitrary in $\mathcal{C}_a(t, x)$. Then, by similar arguments as in 1), we obtain

$$\begin{aligned} \hat{w}(t, x, a) &\geq \int_t^{\infty} e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int w(Y_s^{t,x} + az) p(s, dz) \right] ds \\ &\geq \int_t^{\infty} e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int v(Y_s^{t,x} + az) p(s, dz) \right] ds, \end{aligned}$$

where we used in the second inequality the fact that $w \geq v$. From the arbitrariness of c, we conclude that $\hat{w} \geq \hat{v}$. □

Corollary 3.1. Under **(H1)**, **(H2)**, **(H3)**, (2.4), and (3.12), there exists some positive constant K s.t.

$$\hat{v}(t, x, a) \quad \leq \quad K(e^{bt} x)^{\gamma}, \quad \forall (t, x, a) \in \mathcal{D}, \tag{3.21}$$

$$v(x) \quad \leq \quad K x^{\gamma}, \quad \forall x \geq 0. \tag{3.22}$$

Proof. For ρ large enough, actually satisfying (3.12), we claim that one may find some constants $K \geq 0$ and β s.t.

$$\hat{w}(t, x, a) \quad = \quad K e^{\beta t} x^{\gamma}, \quad (t, x, a) \in \mathcal{D}, \tag{3.23}$$

$$w \quad = \quad \mathcal{H}\hat{w}, \tag{3.24}$$

satisfies (3.17)-(3.18). Indeed, similarly as in (3.15), we notice from Jensen's inequality and conditions **(H2)-(H3)** that for all $(t,x,a) \in \mathcal{D}$,

$$\int (x+az)^{\gamma} p(t,dz) \;\; \leq \;\; \big(x + a\int zp(t,dz)\big)^{\gamma} \; \leq \; x^{\gamma}\frac{\kappa^{\gamma}}{\underline{z}^{\gamma}} e^{b\gamma t}. \tag{3.25}$$

Then, with this choice of $\hat{w}$, noting that $w(x) = Kx^{\gamma}$, and recalling (2.6), we have for all $(t,x,a) \in \mathcal{D}$,

$$\begin{aligned}
& (\rho+\lambda)\hat{w} - \frac{\partial \hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{w}}{\partial x}\right) - \lambda \int w(x+az)p(t,dz) \\
\geq \;\; & Ke^{\beta t}x^{\gamma}(\rho+\lambda-\beta) - \tilde{K}_1(K\gamma e^{\beta t}x^{\gamma-1})^{-\tilde{\gamma}} - \lambda K \int (x+az)^{\gamma} p(t,dz) \\
\geq \;\; & x^{\gamma}\left[K(\rho+\lambda-\beta)e^{\beta t} - K^{-\tilde{\gamma}}\tilde{K}_1\gamma^{-\tilde{\gamma}}e^{-\beta\tilde{\gamma}t} - \lambda K\frac{\kappa^{\gamma}}{\underline{z}^{\gamma}}e^{b\gamma t}\right].
\end{aligned} \tag{3.26}$$

By choosing $\beta = b\gamma$, we then get

$$\begin{aligned}
& (\rho+\lambda)\hat{w} - \frac{\partial \hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{w}}{\partial x}\right) - \lambda \int w(x+az)p(t,dz) \\
\geq \;\; & (Ke^{\beta t})^{-\tilde{\gamma}}x^{\gamma}\left[K^{\frac{1}{1-\gamma}}\left(\rho - b\gamma + \lambda - \frac{\lambda\kappa^{\gamma}}{\underline{z}^{\gamma}}\right)e^{\frac{b\gamma}{1-\gamma}t} - \tilde{K}_1\gamma^{-\tilde{\gamma}}\right]
\end{aligned}$$

Therefore, under (3.12), and by taking K positive s.t.

$$K^{\frac{1}{1-\gamma}}\left(\rho - b\gamma + \lambda - \frac{\lambda\kappa^{\gamma}}{\underline{z}^{\gamma}}\right) \;\; \geq \;\; \tilde{K}_1\gamma^{-\tilde{\gamma}}, \tag{3.27}$$

the pair of functions $(\hat{w}, w)$ defined in (3.23)-(3.24) is a supersolution to (3.17)-(3.18), satisfying the growth condition (3.19). We conclude with Proposition 3.1. □

Remark 3.4. 1) In the case of Example 2.1, we have $\underline{z} = 1$ and $\kappa = 1$. Hence, from (3.12) and (3.27), we may take ρ and K large enough but independently of λ so that $v(x) \leq Kx^{\gamma}$ for all $x \geq 0$. We then have a bound on v uniformly with respect to the intensity λ of the Poisson process. This is important once we want to study the asymptotic analysis of v when λ goes to infinity.
2) Similarly as in Remark 3.3, in the case where $\int zp(t,dz) \leq 0$, and by assuming $-\int zp(t,dz)$ $\leq \kappa e^{bt}$ for some κ, $b \in \mathbb{R}_+$, the inequality (3.25) should be replaced by :

$$\int (x+az)^{\gamma} p(t,dz) \;\; \leq \;\; \big(x + a\int zp(t,dz)\big)^{\gamma} \; \leq \; x^{\gamma}\big(1 + \frac{\kappa^{\gamma}}{\bar{z}^{\gamma}} e^{b\gamma t}\big).$$

Hence, by same arguments as above, we obtain the growth condition (3.21)-(3.22) provided that ρ satisfies (3.16) and with K s.t.

$$K^{\frac{1}{1-\gamma}}\left(\rho - b\gamma - \frac{\lambda\kappa^{\gamma}}{\bar{z}^{\gamma}}\right) \;\; \geq \;\; \tilde{K}_1\gamma^{-\tilde{\gamma}},$$

The point is that in this case ρ and K have to be chosen large enough, depending on λ.

We now state the verification theorem for the coupled IPDE (3.9)-(3.11).

Theorem 3.1. Under **(H1)-(H2)-(H3)**, (2.4) and (3.12), suppose there exist $\hat{w} \in C_+(\mathcal{D})$, C^1 with respect to (t,x), and $w \in C_+(\mathbb{R}_+)$ solution to

$$(\rho+\lambda)\hat{w} - \frac{\partial \hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{w}}{\partial x}\right) - \lambda \int w(x+az)p(t,dz) = 0, \quad (t,x,a) \in \mathcal{D}, \tag{3.28}$$

$$w = \mathcal{H}\hat{w}, \tag{3.29}$$

and that $\hat{w}$ satisfies the growth condition :

$$\hat{w}(t,x,a) \leq K(e^{bt}x)^\gamma, \quad \forall (t,x,a) \in \mathcal{D}, \tag{3.30}$$

for some positive constant K. Consider the nonnegative measurable function

$$\hat{c}(t,x,a) = I\left(\frac{\partial \hat{w}}{\partial x}(t,x,a)\right) = \arg\max_{c \geq 0} \left[U(c) - c\frac{\partial \hat{w}}{\partial x}(t,x,a)\right],$$

and suppose that for all $(t,x,a) \in \mathcal{D}$, there exists a solution, denoted $\hat{Y}_s(t,x,a)$, $s \geq t$, to :

$$dY_s = -\hat{c}(s-t, Y_s, a)ds, \quad s \geq t, \ Y_t = x,$$

s.t. $(s, \hat{Y}_s(t,x,a), a) \in \mathcal{D}$, for $s \geq t$. Then, we have

$$v = w,$$

and an optimal control policy for $v(x)$ is given by :

$$\alpha^*_{k+1} \in \arg \max_{-\frac{X^x_k}{\bar{z}} \leq a \leq \frac{X^x_k}{\underline{z}}} \hat{w}(0, X^x_k, a), \quad k \geq 0, \tag{3.31}$$

$$c^*_t = \hat{c}(t-\tau_k, \hat{Y}_t(\tau_k, X^x_k, \alpha^*_{k+1}), \alpha^*_{k+1}), \quad \tau_k < t \leq \tau_{k+1}. \tag{3.32}$$

Proof. Notice that, since $\tilde{U}(\frac{\partial \hat{w}}{\partial x}) < \infty$, then $\frac{\partial \hat{w}}{\partial x}$ is nonnegative, i.e. $\hat{w}$ is nondecreasing in x. Given $x \geq 0$, consider the control policy (α^*, c^*) defined by (3.31)-(3.32). By construction, the associated wealth process satisfies for all $k \geq 0$,

$$\begin{aligned} X^x_{k+1} &= X^x_k - \int_{\tau_k}^{\tau_{k+1}} c^*_t dt + \alpha^*_{k+1} Z_{k+1} \\ &= \hat{Y}_{\tau_{k+1}}(\tau_k, X^x_k, \alpha^*_{k+1}) + \alpha^*_{k+1} Z_{k+1} \\ &\geq \ell(\alpha^*_{k+1}) + \alpha^*_{k+1} Z_{k+1} \geq 0, \quad a.s. \end{aligned}$$

since $-\underline{z} \leq Z_{k+1} \leq \bar{z}$ a.s. Hence, $(\alpha^*, c^*) \in \mathcal{A}(x)$. Set $\hat{Y}^{(k)}_s = \hat{Y}_s(\tau_k, X^x_k, \alpha^*_{k+1})$, and apply now standard differential calculus to $e^{-(\rho+\lambda)(s-\tau_k)}\hat{w}(s-\tau_k, \hat{Y}^{(k)}_s, \alpha^*_{k+1})$ between $s = \tau_k$ and T (to be sent to infinity) :

$$\begin{aligned} & e^{-(\rho+\lambda)(T-\tau_k)}\hat{w}(T-\tau_k, \hat{Y}^{(k)}_T, \alpha^*_{k+1}) \qquad (3.33) \\ = \ & \hat{w}(0, X^x_k, \alpha^*_{k+1}) \\ & + \int_{\tau_k}^T e^{-(\rho+\lambda)(s-\tau_k)} \left[-(\rho+\lambda)\hat{w} + \frac{\partial \hat{w}}{\partial t} - \hat{c}(s-\tau_k, \hat{Y}^{(k)}_s, \alpha^*_{k+1})\frac{\partial \hat{w}}{\partial x}\right](s-\tau_k, \hat{Y}^{(k)}_s, \alpha^*_{k+1})ds \\ = \ & \hat{w}(0, X^x_k, \alpha^*_{k+1}) \\ & - \int_{\tau_k}^T e^{-(\rho+\lambda)(s-\tau_k)} \left[U(\hat{c}(s-\tau_k, \hat{Y}^{(k)}_s, \alpha^*_{k+1})) + \lambda \int w(\hat{Y}^{(k)}_s + \alpha^*_{k+1} z)p(s-\tau_k, dz)\right] ds, \end{aligned}$$

from (3.28), and since by (2.5)

$$
\begin{aligned}
& \tilde{U}\left(\frac{\partial \hat{w}}{\partial x}(s-\tau_k, \hat{K}_s^{(k)}, \alpha_{k+1}^*)\right) \\
= \;& U(\hat{c}(s-\tau_k, \hat{Y}_s^{(k)}, \alpha_{k+1}^*)) - \hat{c}(s-\tau_k, \hat{Y}_s^{(k)}, \alpha_{k+1}^*)\frac{\partial \hat{w}}{\partial x}(s-\tau_k, \hat{Y}_s^{(k)}, \alpha_{k+1}^*).
\end{aligned}
$$

Now, from the growth condition (3.30) and since $\hat{w}$ is nondecreasing in x, we have

$$
0 \;\leq\; \hat{w}(T-\tau_k, \hat{Y}_T^{(k)}, \alpha_{k+1}^*) \;\leq\; \hat{w}(T-\tau_k, X_k^x, \alpha_{k+1}^*) \;\leq\; K(e^{bT}X_k^x)^\gamma \quad a.s.
$$

from which we deduce by (3.12) that

$$
\lim_{T\to\infty} e^{-(\rho+\lambda)(T-\tau_k)}\hat{w}(T-\tau_k, \hat{Y}_T^{(k)}, \alpha_{k+1}^*) \;=\; 0, \quad a.s.
$$

Hence, by sending T to infinity into (3.33), we obtain by definition of α_{k+1}^*, c^* and w :

$$
\begin{aligned}
w(X_k^x) &= \hat{w}(0, X_k^x, \alpha_{k+1}^*) \\
&= \int_{\tau_k}^{T} e^{-(\rho+\lambda)(s-\tau_k)}\left[U(\hat{c}(s-\tau_k, \hat{Y}_s^{(k)}, \alpha_{k+1}^*)) + \lambda\int w(\hat{Y}_s^{(k)} + \alpha_{k+1}^* z)p(s-\tau_k, dz)\right] ds \\
&= \mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)}U(c_t^*)dt + e^{-\rho(\tau_{k+1}-\tau_k)}w(X_{k+1}^x)\middle| \mathcal{G}_{\tau_k}\right], \quad k \geq 0,
\end{aligned}
$$

by Lemma 3.1. By iterating these relations for all k, and using the law of conditional toy expectations, we obtain

$$
w(x) \;=\; \mathbb{E}\left[\int_0^{\tau_n} e^{-\rho t}U(c_t^*)dt + e^{-\rho\tau_n}w(X_n^x)\right],
$$

for all n. Similarly as in (3.20), we obtain by sending n to infinity :

$$
w(x) \;=\; \mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t^*)dt\right],
$$

which provides the required result, since we already knowed from Proposition 3.1 that $w \geq v$. □

It is an open question to know whether there exists a smooth solution to the IPDE (3.28). To overcome this point, we shall prove in the next section a characterization of the value function by means of viscosity solutions to this IPDE.

4 Viscosity characterization

We adapt now the notion of viscosity solutions to our context, i.e. for the coupled IPDE :

$$
(\rho+\lambda)\hat{w} - \frac{\partial \hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{w}}{\partial x}\right) - \lambda\int w(x+az)p(t,dz) \;=\; 0, \quad (t,x,a) \in \mathcal{D}, \tag{4.1}
$$

$$
w \;=\; \mathcal{H}\hat{w}. \tag{4.2}
$$

Definition 4.1. A pair of functions $(w, \hat{w}) \in C_+(\mathbb{R}_+) \times C_+(\mathcal{D})$ is a viscosity solution to (4.1)-(4.2) if :
(i) *viscosity supersolution property* : $w \geq \mathcal{H}\hat{w}$, and for all $a \in A$,

$$(\rho+\lambda)\hat{w}(\bar{t},\bar{x},a) - \frac{\partial \varphi}{\partial t}(\bar{t},\bar{x}) - \tilde{U}\left(\frac{\partial \varphi}{\partial x}(\bar{t},\bar{x})\right) - \lambda \int w(\bar{x}+az)p(\bar{t},dz) \quad \geq \quad 0,$$

for any test function $\varphi \in C^1(\mathbb{R}_+ \times (\ell(a),\infty))$, and $(\bar{t},\bar{x}) \in \mathbb{R}_+ \times (\ell(a),\infty)$, which is a local minimum of $(\hat{w}(.,.,a) - \varphi)$.
(ii) *viscosity subsolution property* : $w \leq \mathcal{H}\hat{w}$, and for all $a \in A$,

$$(\rho+\lambda)\hat{w}(\bar{t},\bar{x},a) - \frac{\partial \varphi}{\partial t}(\bar{t},\bar{x}) - \tilde{U}\left(\frac{\partial \varphi}{\partial x}(\bar{t},\bar{x})\right) - \lambda \int w(\bar{x}+az)p(\bar{t},dz) \quad \leq \quad 0,$$

for any test function $\varphi \in C^1(\mathbb{R}_+ \times (\ell(a),\infty))$, and $(\bar{t},\bar{x}) \in \mathbb{R}_+ \times (\ell(a),\infty)$, which is a local maximum of $(\hat{w}(.,.,a) - \varphi)$.

Our second main result is a viscosity characterization of the value functions to our control problem. We first prove the continuity of v and $\hat{v}$, and in particular the boundary condition imposed by the state constraint (3.7).

Lemma 4.1. Assume that **(H1)-(H2)-(H3)**, (2.4), and (3.12) hold. The value function v is nondecreasing, concave and continuous on $\mathbb{R}_+$, with $v(0) = 0$.

Proof. Notice that for any $0 \leq x \leq x'$, and any given mixed control (α, c), we have $X_k^x \leq X_k^{x'}$, $k \geq 1$. This implies $\mathcal{A}(x) \subset \mathcal{A}(x')$ and so the nondecreasing property of v. The concavity property of v also follows by standard arguments using the linearity of X_k^x on x, (α, c), and the concavity of U.

Moreover, since v is finite on $\mathbb{R}_+$, it is continuous on $(0,\infty)$. Observe also from the growth condition (3.22) on the nonnegative value function v, that $v(0^+) = 0 = v(0)$. This shows that v is also continuous on $x = 0$. □

Lemma 4.2. Under **(H1)-(H2)-(H3)**, (2.4), and (3.12), the value function $\hat{v}$ defined in (3.6) is continuous on $\mathcal{D}$, and

$$\hat{v}(t,x,a) \quad = \quad \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int v(x+az)p(s,dz)ds, \quad \forall t \geq 0, \ \forall (x,a) \in \partial\mathcal{X}. \tag{4.3}$$

Proof. 1) (i) We first prove the concavity of $\hat{v}(t,.,.)$ in $(x,a) \in \mathcal{X}$ for any $t \in \mathbb{R}_+$. Indeed, this follows from the linearity of the dynamics $Y^{t,x}$ in (3.8) in x, the linearity in (x,a) of the admissibility condition (3.7), and the concavity of the reward functions U and v appearing in the definition (3.6) of $\hat{v}$. Since we have also showed in (3.21) that $\hat{v}$ is finite on $\mathcal{D}$, this implies the continuity of v on the interior int$(\mathcal{X})$ of $\mathcal{X}$.
(ii) We now show the continuity of $\hat{v}$ on $\partial\mathcal{X}$. Fix some $t \in \mathbb{R}_+$, and take some $(x_0, a_0) \in \partial\mathcal{X}$, i.e. $a_0 \in A$ and $x_0 = \ell(a_0)$. Since $\mathcal{C}_{a_0}(t,x_0) = \{0\}$ by (3.7), we have

$$\hat{v}(t,x_0,a_0) \quad = \quad \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int v(x_0+a_0 z)p(s,dz)ds. \tag{4.4}$$

Fix now some arbitrary $\varepsilon > 0$. By continuity of the function $a \in A \mapsto \ell(a)$, one can find some $\delta > 0$ s.t. for all $(x,a) \in \mathcal{X}_\delta = \{(x,a) \in \mathcal{X} \ : |x-x_0| + |a-a_0| < \delta\}$, we have $x - \ell(a) < \varepsilon^{1+1/\tilde{\gamma}}$, and so by (3.7)

$$\int_t^\infty c_s ds \quad < \quad \varepsilon^{1+\frac{1}{\tilde{\gamma}}}, \quad \forall\, c \in \mathcal{C}_a(t,x), \tag{4.5}$$

where $\tilde\gamma$ was defined in (2.6). Now, by choosing y s.t. $\tilde K_1 y^{-\tilde\gamma} = \varepsilon$ in (2.6), we have for any $c \geq 0$, $U(c) \leq \tilde U(y) + cy \leq \varepsilon + c(\tilde K_1/\varepsilon)^{\frac{1}{\tilde\gamma}}$. Hence, for all $(x,a) \in \mathcal{X}_\delta$, we have

$$\begin{aligned}\int_t^\infty e^{-(\rho+\lambda)(s-t)} U(c_s) ds &\leq \int_t^\infty e^{-(\rho+\lambda)(s-t)} \left(\varepsilon + c_s(\tilde K_1/\varepsilon)^{\frac{1}{\tilde\gamma}}\right) ds \\ &\leq \varepsilon\left(\frac{1}{\rho+\lambda} + \tilde K_1^{\frac{1}{\tilde\gamma}}\right), \quad \forall\, c \in \mathcal{C}_a(t,x),\end{aligned}$$

by (4.5). We deduce for all $(x,a) \in \mathcal{X}_\delta$:

$$\begin{aligned}&|\hat v(t,x,a) - \hat v(t,x_0,a_0)| \\ \leq\ & \sup_{c \in \mathcal{C}_a(t,x)} \Bigg[\int_t^\infty e^{-(\rho+\lambda)(s-t)} U(c_s) ds \\ &\qquad + \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int |v(Y_s^{t,x} + az) - v(x_0 + a_0 z)| p(s,dz) ds\Bigg] \\ \leq\ & \varepsilon\left(\frac{1}{\rho+\lambda} + \tilde K_1^{\frac{1}{\tilde\gamma}}\right) \\ &+ \sup_{c\in\mathcal{C}_a(t,x)} \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int |v(Y_s^{t,x} + az) - v(x_0 + a_0 z)| p(s,dz) ds \qquad (4.6)\end{aligned}$$

$\star$ Consider first the case where $\bar z < \infty$. By noting that for all $c \in \mathcal{C}_a(t,x)$, $s \geq t$, $|Y_s^{t,x} - x_0| \leq |x - x_0| + |x - \ell(a)|$, and by continuity of the function v, one may still choose $\delta > 0$ small enough so that for all $(x,a) \in \mathcal{X}_\delta$:

$$\sup_{z \in [-\underline{z}, \bar z]} \left|v(Y_s^{t,x} + az) - v(x_0 + a_0 z)\right| \ \leq\ \varepsilon, \quad \forall s \geq t,\ \forall c \in \mathcal{C}_a(t,x). \qquad (4.7)$$

By plugging into (4.6), we obtain for all $(x,a) \in \mathcal{X}_\delta$:

$$|\hat v(t,x,a) - \hat v(t,x_0,a_0)| \ \leq\ \varepsilon\left(\frac{1+\lambda}{\rho+\lambda} + \tilde K_1^{\frac{1}{\tilde\gamma}}\right), \qquad (4.8)$$

which proves the continuity of $\hat v$ on (t,x_0,a_0).

$\star$ Consider now the case where $\bar z = \infty$. Then $A = \mathbb{R}_+$ and $x_0 = a_0 \underline{z}$. Suppose that $a_0 = 0$, and so $\hat v(t,x_0,a_0) = 0$ by (4.4). Recalling that v is nondecreasing, and from the growth condition (3.22) on v together with (3.25), we have for all $(x,a) \in \mathcal{X}_\delta$:

$$\begin{aligned}\int_{-\underline{z}}^\infty v(Y_s^{t,x} + az) p(s,dz) &\leq \int_{-\underline{z}}^\infty v(x + az) p(s,dz) \ \leq\ K x^\gamma \frac{\kappa^\gamma}{\underline{z}^\gamma} e^{b\gamma s} \\ &\leq K\varepsilon\, e^{b\gamma s}, \quad \forall s \geq t,\ \forall c \in \mathcal{C}_a(t,x),\end{aligned}$$

by choosing δ s.t. $(\delta\kappa/\underline{z})^\gamma < \varepsilon$. By plugging into (4.6), we obtain for all $(x,a) \in \mathcal{X}_\delta$:

$$|\hat v(t,x,a) - \hat v(t,0,0)| \ \leq\ \varepsilon\left(\frac{1}{\rho+\lambda} + \tilde K_1^{\frac{1}{\tilde\gamma}} + \frac{\lambda K e^{b\gamma t}}{\rho + \lambda - b\gamma}\right), \qquad (4.9)$$

which proves the continuity of $\hat v$ on $(t,0,0)$. Suppose $a_0 > 0$, i.e. $x_0 > 0$, so that w.l.o.g. we may assume that $\delta + \varepsilon^{1+1/\tilde\gamma} < x_0/2$. Hence, for all $(x,a) \in \mathcal{X}_\delta$, $c \in \mathcal{C}_a(t,x)$, we have by (4.5), $Y_s^{t,x} + az \geq x_0/2$, for any $t \leq s$, $z \geq 0$. Moreover, since the function v is concave and finite on $\mathbb{R}_+$, it is Lipschitz on $[x_0/2, \infty)$. Thus, for all $(x,a) \in \mathcal{X}_\delta$, $c \in \mathcal{C}_a(t,x)$, there exists some positive constant C_0 s.t.

$$\begin{aligned}
\left|v(Y_s^{t,x} + az) - v(x_0 + a_0 z)\right| &\leq C_0 \left(|Y_s^{t,x} - x_0| + |a - a_0| z\right) \\
&\leq C_0 \left(\delta + \varepsilon^{1+\frac{1}{\gamma}} + \delta z\right) \\
&\leq C_0 \varepsilon (2 + z), \quad t \leq s, \; z \geq 0, \qquad (4.10)
\end{aligned}$$

for $0 < \delta < \varepsilon < 1$. On the other hand, similarly as in (4.7), we have

$$\sup_{z \in [-\underline{z}, 0]} \left|v(Y_s^{t,x} + az) - v(x_0 + a_0 z)\right| \leq \varepsilon, \quad \forall s \geq t, \; \forall c \in \mathcal{C}_a(t,x). \qquad (4.11)$$

By plugging (4.10)-(4.11) into (4.6), we obtain for all $(x,a) \in \mathcal{X}_\delta$:

$$|\hat{v}(t,x,a) - \hat{v}(t,x_0,a_0)| \leq \varepsilon \left(\frac{1 + \lambda + \lambda C_0}{\rho + \lambda} + \tilde{K}_1^{\frac{1}{\gamma}} + \frac{\lambda C_0 \kappa e^{bt}}{\rho + \lambda - b} \right), \qquad (4.12)$$

which proves the continuity of $\hat{v}$ on (t, x_0, a_0).

2) We next prove the continuity of $\hat{v}(t,x,a)$ in $t \in \mathbb{R}_+$ for fixed $(x,a) \in \mathcal{X}$. Fix some arbitrary $\varepsilon > 0$. Since $x - Y_s^{t,x} = \int_t^s c_u du \leq x - \ell(a)$ for all $t \leq s$, $c \in \mathcal{C}_a(t,x)$, one can find $0 < \delta < \varepsilon$ s.t. for all $0 \leq t < s$, $|s - t| < \delta$, we have

$$\begin{aligned}
\int_t^s c_u du &\leq \varepsilon, \quad \forall c \in \mathcal{C}_a(t,x) \qquad (4.13) \\
|\hat{v}(s, Y_s^{t,x}, a) - \hat{v}(s,x,a)| &\leq \varepsilon, \quad \forall c \in \mathcal{C}_a(t,x). \qquad (4.14)
\end{aligned}$$

Now, from the dynamic programming principle applied to $\hat{v}(t,x,a)$, we have for all $s \geq t$:

$$\begin{aligned}
\hat{v}(t,x,a) = \sup_{c \in \mathcal{C}_a(t,x)} \Big\{ &\int_t^s e^{-(\rho+\lambda)(u-t)} \left[U(c_u) + \lambda \int v(Y_u^{t,x} + az) p(u,dz) \right] du \\
&+ e^{-(\rho+\lambda)(s-t)} \hat{v}(s, Y_s^{t,x}, a) \Big\}.
\end{aligned}$$

Recalling that v is nondecreasing, this yields for all $0 \leq t < s$, $|s - t| < \delta$:

$$\begin{aligned}
|\hat{v}(t,x,a) - \hat{v}(s,x,a)| &\leq \sup_{c \in \mathcal{C}_a(t,x)} \Big\{ \int_t^s e^{-(\rho+\lambda)(u-t)} \left[U(c_u) + \lambda \int v(x + az) p(u,dz) \right] du \\
&\qquad + e^{-(\rho+\lambda)(s-t)} \hat{v}(s, Y_s^{t,x}, a) - \hat{v}(s,x,a) \Big\} \\
&\leq \sup_{c \in \mathcal{C}_a(t,x)} \Big\{ \int_t^s \left[c_u + \tilde{U}(1) + \lambda K x^\gamma \frac{\kappa^\gamma e^{bs}}{\underline{z}^\gamma} \right] du \\
&\qquad + |\hat{v}(s, Y_s^{t,x}, a) - \hat{v}(s,x,a)| + (\rho + \lambda)|s - t| \hat{v}(s,x,a) \Big\} \\
&\leq \varepsilon \left(2 + \tilde{U}(1) + \lambda K x^\gamma \frac{\kappa^\gamma e^{bs}}{\underline{z}^\gamma} + (\rho + \lambda) K x^\gamma e^{b\gamma s} \right), \qquad (4.15)
\end{aligned}$$

where we used in the second inequality the relation $U(c) \leq c + \tilde{U}(1)$ for all $c \geq 0$, the growth condition (3.22) on v and (3.25), and in the last inequality the relations (4.13)-(4.14). This proves the continuity of $\hat{v}$ in t.

3) Finally, by combining inequalities (4.8), (4.9), (4.12), and (4.15), we have the continuity of $\hat{v}$ in $(t,x,a) \in \mathcal{D}$. $\square$

Remark 4.1. The arguments in the above Lemma for proving the continuity of $\hat{v}$ on the boundary $\partial\mathcal{X}$ show that this boundary is absorbing : indeed, when $(x,a) \in \partial\mathcal{X}$, i.e. $x = \ell(a)$, the only admissible control for $c \in \mathcal{C}_a(t,x)$ is $c = 0$, so that the state process $Y^{t,x}$ remains at $\ell(a)$ once it reaches this threshold.

Remark 4.2. Notice from (4.3) that $\hat{v}$ is differentiable in t for $(x,a) \in \partial\mathcal{X}$, and so this boundary condition may be also formulated as :

$$\begin{aligned} \lim_{t\to\infty} e^{-(\rho+\lambda)t}\hat{v}(t,x,a) &= 0, \quad \forall (x,a) \in \partial\mathcal{X}, \\ (\rho+\lambda)\hat{v}(t,x,a) - \frac{\partial \hat{v}}{\partial t}(t,x) - \lambda \int v(x+az)p(t,dz) &= 0, \ \forall t \geq 0, \forall (x,a) \in \partial\mathcal{X}. \end{aligned} \tag{4.16}$$

We now provide a complete characterization of the value function of our original control problem by means of viscosity solution to the coupled IPDE. This is achieved in two steps. We first prove, as usual, the viscosity property as a consequence of the dynamic programming principle. We then prove a new comparison principle for the coupled IPDE (4.1)-(4.2). We make an additional continuity assumption on the measure $p(t,dz)$:

$$\lim_{t\to t_0} \int w(z)p(t,dz) = \int w(z)p(t_0,dz), \quad \forall t_0 \geq 0, \tag{4.17}$$

for all measurable functions w on $(-\underline{z}, \bar{z})$ with linear growth condition.

Theorem 4.1. Assume that **(H1)-(H2)-(H3)**, (2.4), (3.12) and (4.17) hold. The pair of value functions $(v,\hat{v})$ defined in (2.3)-(3.6) is the unique viscosity solution to (4.1)-(4.2), satisfying the growth condition (3.21)-(3.22), and the boundary condition (4.3).

Proof. 1) From the dynamic programming principle (3.1) proved in Appendix, and following the arguments in (3.2)-(3.9), we prove that $v = \mathcal{H}\hat{v}$. Moreover, for each $a \in A$, $\hat{v}(.,.,a)$ is the value function of a deterministic time-dependent control problem with state Y. Hence, standard dynamic programming principle in this context, see e.g. Fleming and Soner [6], yields the viscosity property of $\hat{v}(.,.,a)$ to (4.1), and so the viscosity property of $(v,\hat{v})$ to (4.1)-(4.2). The growth condition (3.21)-(3.22), and the boundary condition (4.3) are proved in Corollary 3.1 and Lemma 4.2.
2) The main task is to prove the following comparison principle : if $(w_1,\hat{w}_1)$ (resp. $(w_2,\hat{w}_2)$) $\in C_+(\mathbb{R}_+)\times C_+(\mathcal{D})$ is a viscosity subsolution (resp. supersolution) to (4.1)-(4.2), satisfying the growth condition (3.21)-(3.22), and :

$$\hat{w}_1(t,x,a) = \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int w_1(x+az)p(s,dz)ds, \quad \forall t \geq 0, \ \forall (x,a) \in \partial\mathcal{X}$$

then $w_1 \leq w_2$, and $\hat{w}_1 \leq \hat{w}_2$. Uniqueness result is then a direct corollary.
Step 1. In a first step, we deal with the noncompactness of the domain (regarding the growth condition of $\hat{w}_1, \hat{w}_2$ in (t,x)) by constructing a suitable perturbation of the viscosity supersolution $(w_2,\hat{w}_2)$. Under (3.12), we can choose $\gamma' \in (\gamma,1)$, and $\rho' > 0$ s.t.

$$b\gamma' < \rho' \leq \rho - \lambda\Big(\frac{\kappa^{\gamma'}}{\underline{z}^{\gamma'}} - 1\Big) \tag{4.18}$$

Now, let for all $n \geq 1$, $\hat{w}_{2,n} = \hat{w}_2 + \frac{1}{n}\hat{\psi}$, $w_{2,n} = w_2 + \frac{1}{n}\psi$, , with $\hat{\psi}(t,x) = e^{\rho' t}x^{\gamma'}$ and $\psi(x) = \mathcal{H}\hat{\psi}(x) = x^{\gamma'}$. From condition **(H3)**, and by similar calculations as in (3.26), we see that for all $(t,x,a) \in \mathcal{D}$,

$$\begin{aligned}
& (\rho+\lambda)\hat{\psi} - \frac{\partial \hat{\psi}}{\partial t} - \lambda \int \psi(x+az)p(t,dz) \\
\geq \;\; & x^{\gamma'}\left[(\rho+\lambda-\rho')e^{\rho' t} - \lambda \frac{\kappa^{\gamma'} e^{b\gamma' t}}{\underline{z}^{\gamma'}}\right] \\
\geq \;\; & x^{\gamma'} e^{\rho' t}\left[\rho - \rho' + \lambda - \lambda \frac{\kappa^{\gamma'}}{\underline{z}^{\gamma'}}\right] \;\geq\; 0,
\end{aligned}$$

by (4.18). By noting also that $\tilde{U}$ is nonincreasing, we then deduce that $(w_{2,n}, \hat{w}_{2,n})$ is a viscosity supersolution to (4.1)-(4.2). Moreover, from the growth condition (3.21) on $\hat{w}_1$, and $\hat{w}_2$, and since $\gamma' > \gamma$, $\rho' > b\gamma'$, we have for all $n \geq 1$:

$$\lim_{|(t,x)| \to \infty} \sup_{a \in A} (\hat{w}_1 - \hat{w}_{2,n})(t,x,a) \;\; = \;\; -\infty. \tag{4.19}$$

Step 2. We show that for all $n \geq 1$, $\hat{w}_1 \leq \hat{w}_{2,n}$ on $\mathcal{D}$. We argue by contradiction, and assume on the contrary that there exists some $n \geq 1$ s.t.

$$M \;\; := \;\; \sup_{(t,x,a) \in \mathcal{D}} (\hat{w}_1 - \hat{w}_{2,n})(t,x,a) \; > \; 0.$$

In this case, from (4.19) and by continuity of $\hat{w}_1$ and $\hat{w}_2$, there exists some compact subset $\mathcal{D}_0$ of $\mathcal{D}$, which may be chosen in the form $\mathcal{D}_0 = [0,T_0] \times \mathcal{X}_0$ with

$$\mathcal{X}_0 \;\; = \;\; \{(x,a) \in \mathcal{X} \; : x \leq x_0\} \;\; = \;\; \left\{(x,a) \in \mathbb{R}_+ \times \left[-\frac{x_0}{\bar{z}}, \frac{x_0}{\underline{z}}\right] \;\; : x \in [\ell(a), x_0]\right\}$$

for some finite positive $T_0 > 0$ and $x_0 > 0$ (depending on n), and $(\bar{t}, \bar{x}, \bar{a}) \in \mathcal{D}_0$ with $\bar{t} < T_0$, $\bar{x} < x_0$ s.t.

$$M \;\; = \;\; \max_{(t,x,a) \in \mathcal{D}_0} (\hat{w}_1 - \hat{w}_{2,n})(t,x,a) \;\; = \;\; (\hat{w}_1 - \hat{w}_{2,n})(\bar{t}, \bar{x}, \bar{a})$$

We distinguish the two cases depending on $(\bar{x}, \bar{a}) \in \partial\mathcal{X}$, i.e. $\bar{x} = \ell(\bar{a})$, or $(\bar{x}, \bar{a}) \notin \partial\mathcal{X}$, i.e. $\bar{x} > \ell(\bar{a})$.

$\star$ *Case 1.* : $\bar{x} > \ell(\bar{a})$.

Following the general technique for comparison principle, we then consider, for any $\varepsilon > 0$, the function defined by

$$\begin{aligned}
\Phi_\varepsilon(t,s,x,y) \;\; &= \;\; \hat{w}_1(t,x,\bar{a}) - \hat{w}_{2,n}(s,y,\bar{a}) - \phi_\varepsilon(t,s,x,y) \\
\phi_\varepsilon(t,s,x,y) \;\; &= \;\; \frac{|t-\bar{t}|^2}{2} + \frac{|x-\bar{x}|^3}{3} + \frac{|t-s|^2}{2\varepsilon} + \frac{|x-y|^2}{2\varepsilon}.
\end{aligned} \tag{4.20}$$

Since Φ_ε is continuous on the compact set $[0,T_0]^2 \times [\ell(\bar{a}), x_0]^2$, there exists $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0,T_0]^2 \times [\ell(\bar{a}), x_0]^2$ s.t.

$$M_\varepsilon \;\; := \;\; \sup_{[0,T_0]^2 \times [\ell(\bar{a}), x_0]^2} \Phi_\varepsilon(t,s,x,y) \;\; = \;\; \Phi_\varepsilon(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon),$$

and a subsequence, still denoted $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)_{\varepsilon>0}$, converging to some $(\bar t', \bar s', \bar x', \bar y')$ when ε goes to zero. Actually, by standard arguments in viscosity solutions theory (see e.g. Lemma 2.3 p. 28 in Barles [1]), we have

$$(\bar t', \bar s', \bar x', \bar y') \;=\; (\bar t, \bar t, \bar x, \bar x) \tag{4.21}$$

In particular, for ε small enough, we have $(t_\varepsilon, s_\varepsilon) \in [0, T_0)^2$ and $(x_\varepsilon, y_\varepsilon) \in (\ell(\bar a), x_0)^2$. Hence, Φ_ε admits a local maximum at $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)$. This implies that the function $(t, x) \to \hat w_1(t, x, \bar a) - \varphi_1(t, x)$, with $\varphi_1(t, x) = \frac{|t-\bar t|^2}{2} + \frac{|x-\bar x|^3}{3} + \frac{|t-s_\varepsilon|^2}{2\varepsilon} + \frac{|x-y_\varepsilon|^2}{2\varepsilon}$, admits a local maximum at $(t_\varepsilon, x_\varepsilon)$. By writing the viscosity subsolution property of $(w_1, \hat w_1)$ to (4.1)-(4.2) at $(t_\varepsilon, x_\varepsilon, \bar a)$ with this test function φ_1, we have

$$\begin{aligned}(\rho+\lambda)\hat w_1(t_\varepsilon, x_\varepsilon, \bar a) - (t_\varepsilon - \bar t) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde U\left(|x_\varepsilon - \bar x|^2 + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right)&\\ -\lambda \int w_1(x_\varepsilon + \bar a z) p(t_\varepsilon, dz) \;\le\; 0.&\end{aligned} \tag{4.22}$$

Likewise, the function $(s, y) \to \hat w_{2,n}(s, y, \bar a) - \varphi_2(s, y)$, with $\varphi_2(s, y) = -\frac{|t_\varepsilon - s|^2}{2\varepsilon} - \frac{|x_\varepsilon - y|^2}{2\varepsilon}$, admits a local minimum at $(s_\varepsilon, y_\varepsilon)$. By writing the viscosity supersolution property of $(w_{2,n}, \hat w_{2,n})$ to (4.1)-(4.2) at $(s_\varepsilon, y_\varepsilon, \bar a)$ with this test function φ_2, we have

$$\begin{aligned}(\rho+\lambda)\hat w_{2,n}(s_\varepsilon, y_\varepsilon, \bar a) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde U\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right)&\\ -\lambda \int w_{2,n}(y_\varepsilon + \bar a z) p(s_\varepsilon, dz) \;\ge\; 0.&\end{aligned} \tag{4.23}$$

By substracting (4.22) and (4.23), and since $\tilde U$ is nonincreasing, we obtain

$$\begin{aligned}&(\rho+\lambda)\left(\hat w_1(t_\varepsilon, x_\varepsilon, \bar a) - \hat w_{2,n}(s_\varepsilon, y_\varepsilon, \bar a)\right)\\ \le\;& (t_\varepsilon - \bar t) + \lambda \left[\int w_1(x_\varepsilon + \bar a z) p(t_\varepsilon, dz) - \int w_{2,n}(y_\varepsilon + \bar a z) p(s_\varepsilon, dz)\right].\end{aligned}$$

By sending ε to zero, and from (4.17), (4.21), we get :

$$\begin{aligned}(\rho+\lambda) M &= (\rho+\lambda)(\hat w_1 - \hat w_{2,n})(\bar t, \bar x, \bar a)\\ &\le \lambda \int (w_1 - w_{2,n})(\bar x + \bar a z) p(\bar t, dz)\\ &\le \lambda \int (\mathcal H \hat w_1 - \mathcal H \hat w_{2,n})(\bar x + \bar a z) p(\bar t, dz),\end{aligned}$$

since $w_1 \le \mathcal H \hat w_1$ and $w_{2,n} \ge \mathcal H \hat w_{2,n}$. Finally, by noting from the definition of $\mathcal H$ and M that $\mathcal H \hat w_1 - \mathcal H \hat w_{2,n} \le M$, we get the required contradiction $(\rho+\lambda) M \le \lambda M$.

$\star$ *Case 2.* : $\bar x = \ell(\bar a)$.

Notice that (4.16) implies that the viscosity subsolution property for $(v, \hat v)$ holds also at any $(t, x, a) \in \mathbb R_+ \times \partial \mathcal X$. However, this is not true for the viscosity supersolution property, and we have to modify the test function Φ_ε in (4.20) in order to ensure that for the local minimum point $(s_\varepsilon, y_\varepsilon)$, $(y_\varepsilon, \bar a) \notin \partial \mathcal X$, i.e. $y_\varepsilon > \ell(\bar a)$. We follow arguments in Barles [1]. By continuity of $\hat w_{2,n}$ on $\mathcal D$, there exists a sequence $(\bar t_\varepsilon, \bar x_\varepsilon)_{\varepsilon>0}$, with $\bar x_\varepsilon > \bar x = \ell(\bar a)$, $\bar t_\varepsilon \neq$

$\bar{t}$, converging to $(\bar{t},\bar{x})$ s.t. $\hat{w}_{2,n}(\bar{t}_\varepsilon,\bar{x}_\varepsilon,\bar{a})$ tends to $\hat{w}_{2,n}(\bar{t},\bar{x},\bar{a})$ as ε goes to zero. We then consider the function

$$\begin{aligned}
\Psi_\varepsilon(t,s,x,y) &= \hat{w}_1(t,x,\bar{a}) - \hat{w}_{2,n}(s,y,\bar{a}) - \psi_\varepsilon(t,s,x,y)\\
\psi_\varepsilon(t,s,x,y) &= \frac{|t-\bar{t}|^2}{2} + \frac{|x-\bar{x}|^3}{3} + \frac{|t-s|^2}{2|\bar{t}_\varepsilon - \bar{t}|} + \frac{|x-y|^2}{2|\bar{x}_\varepsilon - \bar{x}|} + \frac{1}{3}\left|\frac{y-\ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1\right|^3.
\end{aligned}$$

Since Ψ_ε is continuous on the compact set $[0,T_0]^2 \times [\ell(\bar{a}),x_0]^2$, there exists $(t_\varepsilon,s_\varepsilon,x_\varepsilon,y_\varepsilon) \in [0,T_0]^2 \times [\ell(\bar{a}),x_0]^2$ s.t.

$$N_\varepsilon := \sup_{[0,T_0]^2\times[\ell(\bar{a}),x_0]^2} \Psi_\varepsilon(t,s,x,y) = \Phi_\varepsilon(t_\varepsilon,s_\varepsilon,x_\varepsilon,y_\varepsilon),$$

and a subsequence, still denoted $(t_\varepsilon,s_\varepsilon,x_\varepsilon,y_\varepsilon)_{\varepsilon>0}$, converging to some $(\bar{t}',\bar{s}',\bar{x}',\bar{y}')$ when ε goes to zero. Again, by standard arguments in viscosity solutions theory, we have

$$\begin{aligned}
(\bar{t}',\bar{s}',\bar{x}',\bar{y}') &= (\bar{t},\bar{t},\bar{x},\bar{x}) && (4.24)\\
N_\varepsilon &\longrightarrow M && (4.25)\\
\frac{|t_\varepsilon - s_\varepsilon|^2}{2|\bar{t}_\varepsilon - \bar{t}|} + \frac{|x_\varepsilon - y_\varepsilon|^2}{2|\bar{x}_\varepsilon - \bar{x}|} + \frac{1}{3}\left|\frac{y_\varepsilon - \ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1\right|^3 &\longrightarrow 0, && (4.26)
\end{aligned}$$

In particular, for ε small enough, we have $(t_\varepsilon,s_\varepsilon) \in [0,T_0)^2$ and $|y_\varepsilon - \ell(\bar{a})| > |\bar{x}_\varepsilon - \ell(\bar{a})|/2 > 0$ and so $y_\varepsilon > \ell(\bar{a})$. We can then write the viscosity subsolution property of $(w_1,\hat{w}_1)$ to (4.1)-(4.2) at $(t_\varepsilon,x_\varepsilon,\bar{a})$ with the test function $(t,x) \mapsto \frac{|t-\bar{t}|^2}{2} + \frac{|x-\bar{x}|^3}{3} + \frac{|t-s_\varepsilon|^2}{2|\bar{t}_\varepsilon - \bar{t}|} + \frac{|x-y_\varepsilon|^2}{2|\bar{x}_\varepsilon-\bar{x}|}$, and the viscosity subsolution property of $(w_{2,n},\hat{w}_{2,n})$ to (4.1)-(4.2) at $(s_\varepsilon,y_\varepsilon,\bar{a})$ with the test function $(s,y) \mapsto -\frac{|t_\varepsilon - s|^2}{2|\bar{t}_\varepsilon - \bar{t}|} - \frac{|x_\varepsilon - y|^2}{2|\bar{x}_\varepsilon - \bar{x}|} - \frac{1}{3}\left|\frac{y-\ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1\right|^3$. This means :

$$\begin{aligned}
&(\rho+\lambda)\hat{w}_1(t_\varepsilon,x_\varepsilon,\bar{a}) - (t_\varepsilon - \bar{t}) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde{U}\left(|x_\varepsilon - \bar{x}|^2 + \frac{x_\varepsilon - y_\varepsilon}{|\bar{x}_\varepsilon - \bar{x}|}\right)\\
&\quad -\lambda\int w_1(x_\varepsilon + \bar{a}z)p(t_\varepsilon,dz) \le 0, \qquad (4.27)
\end{aligned}$$

and

$$\begin{aligned}
&(\rho+\lambda)\hat{w}_{2,n}(s_\varepsilon,y_\varepsilon,\bar{a}) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde{U}\left(\frac{x_\varepsilon - y_\varepsilon}{|\bar{x}_\varepsilon - \bar{x}|} - \frac{1}{\bar{x}_\varepsilon - \ell(\bar{a})}\left|\frac{y-\ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1\right|^2\right)\\
&\quad -\lambda\int w_{2,n}(y_\varepsilon + \bar{a}z)p(s_\varepsilon,dz) \ge 0. \qquad (4.28)
\end{aligned}$$

Again by substracting these two inequalities and since $\tilde{U}$ is nonincreasing, we get

$$\begin{aligned}
&(\rho+\lambda)\left(\hat{w}_1(t_\varepsilon,x_\varepsilon,\bar{a}) - \hat{w}_{2,n}(s_\varepsilon,y_\varepsilon,\bar{a})\right)\\
\le\;& (t_\varepsilon - \bar{t}) + \lambda\left[\int w_1(x_\varepsilon + \bar{a}z)p(t_\varepsilon,dz) - \int w_{2,n}(y_\varepsilon + \bar{a}z)p(s_\varepsilon,dz)\right].
\end{aligned}$$

We then get the required contradiction similarly as in case 1.

Step 3. Now, since $\hat{w}_1 \le \hat{w}_{2,n}$ for all n, we obtain by sending n to infinity : $\hat{w}_1 \le \hat{w}_2$. Therefore, we finally get : $w_1 \le \mathcal{H}\hat{w}_1 \le \mathcal{H}\hat{w}_2 \le w_2$. This ends the proof. □

Remark 4.3. Once we have characterized the value function through its dynamic programming equation by means of viscosity solutions, another question is to characterize the optimal control as in the verification theorem 3.1 when the value function was supposed to be smooth. This can be done with smooth solutions of the dynamic programming equation replaced by viscosity solutions, and derivatives involved replaced by super and subdifferentials, as described in Theorem 3.9 in [15].

5 A numerical decoupling algorithm

The main difficulty in the resolution (both theoretically and numerically) of the IPDE (4.1) for $\hat{v}$ comes from the integrodifferential term involving $\mathcal{H}\hat{v}$. To overcome this problem, we suggest the following iterative procedure. We start from an initial function v_0 defined on $\mathbb{R}_+$, as the value function of the consumption problem without trading :

$$v_0(x) \;=\; \sup_{c \in \mathcal{C}(x)} \int_0^\infty e^{-\rho t} U(c_t) dt,$$

where $\mathcal{C}(x)$ is the set of nonnegative (deterministic) processes $c = (c_t)_t$ s.t. $x - \int_0^t c_s ds \geq 0$ for all $t \geq 0$. v_0 is the unique solution with linear growth condition to the first-order differential equation

$$\rho v_0 - \tilde{U}\left(\frac{\partial v_0}{\partial x}\right) \;=\; 0, \quad x > 0,$$

together with the boundary condition $v_0(0^+) = 0$. We then construct a sequence of functions $(\hat{v}_n(t,x,a))_{n\geq 1}$ defined on $\mathcal{D}$ and $(v_n(x))_{n\geq 0}$ defined on $\mathbb{R}_+$ by :

$$\hat{v}_{n+1}(t,x,a) = \sup_{c \in \mathcal{C}_a(t,x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int v_n(Y_s^{t,x} + az) p(s,dz) \right] ds \tag{5.1}$$

$$v_{n+1} = \mathcal{H}\hat{v}_{n+1}, \qquad n \geq 0.$$

By similar arguments as in the previous section (actually simpler since here there is no more coupling system), one can show that $\hat{v}_{n+1}$ and v_{n+1} are characterized as the unique viscosity solutions to the recursive system :

$$-(\rho+\lambda)\hat{v}_{n+1} + \frac{\partial \hat{v}_{n+1}}{\partial t} + \tilde{U}\left(\frac{\partial \hat{v}_{n+1}}{\partial x}\right) + \lambda \int_{-1}^{\bar{z}} v_n(x+az) p(t,dz) = 0, \quad (t,x,a) \in \mathcal{D}, \tag{5.2}$$

$$v_{n+1} \;=\; \mathcal{H}\hat{v}_{n+1},$$

in the class of functions satisfying the growth condition (3.21)-(3.22), together with the boundary condition :

$$\hat{v}_{n+1}(t,x,a) \;=\; \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int v_n(x+az) p(s,dz) ds, \; \forall t \geq 0, \forall (x,a) \in \partial\mathcal{X}. \tag{5.3}$$

Under suitable conditions, we may expect that the solution $\hat{v}_{n+1}$ to this first-order equation is C^1 so that to obtain an approximate control policy by taking :

$$\alpha_{k+1}^{(n)} \;\in\; \arg \max_{-\frac{X^x_{\tau_k}}{\bar{z}} \leq a \leq \frac{X_{\tau_k}}{\underline{z}}} \hat{v}_n(0, X^x_{\tau_k}, a), \quad k \geq 0,$$

$$c_t^{(n)} \;=\; \hat{c}_n(t - \tau_k, \hat{Y}_t^{(n)}(\tau_k, X^x_{\tau_k}, \alpha^{(n)}_{k+1}), \alpha^{(n)}_{k+1}), \quad \tau_k < t \leq \tau_{k+1}.$$

where

$$\hat{c}_n(t,x,a) \;=\; I\left(\frac{\partial \hat{v}_n}{\partial x}(t,x,a)\right).$$

and $\{\hat{Y}_s^{(n)}(t,x,a),\, t \leq s\}$, is the unique solution to :

$$dY_s \;=\; \hat{c}_n(s-t, Y_s, a)ds, \quad t \leq s,\; Y_t \;=\; x.$$

Notice also that the determination of v_{n+1} is easily obtained from the operator $\mathcal{H}$, involving simply a standard maximization procedure.

5.1 Numerical solution of the decoupled control problem

At step n of the iterative algorithm, to solve the deterministic control problem (5.1) we need to compute $\hat{v}_{n+1}(0,x,a)$ for different values of a (on a discrete grid) and then find the maximum of $\hat{v}_{n+1}(0,x,a)$ to compute $v_{n+1}(x)$. We now explain how the optimization problem (5.1) is solved for each given value of a, and to simplify notation, we drop the dependence on a and write

$$f_n(t,x) \equiv \lambda \int v_n(x+az)p(t,dz).$$

The deterministic control problem to be solved is therefore

$$\hat{v}_{n+1}(t,x) = \sup_{c \in \mathcal{C}_a(t,x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)}[U(c_s) + f_n(s, Y_s^{t,x})]ds.$$

The dynamic programming principle for this control problem implies for $T > t$:

$$\hat{v}_{n+1}(t,x) = \sup_{c \in \mathcal{C}_a(t,x)} \left(\int_t^T e^{-(\rho+\lambda)(s-t)}[U(c_s) + f_n(s, Y_s^{t,x})]ds + e^{-(\rho+\lambda)(T-t)}\hat{v}_{n+1}(T, Y_T^{t,x})\right).$$

Introduce a finite-horizon deterministic control problem

$$\hat{v}_{n+1}^T(t,x) = \sup_{c \in \mathcal{C}_a(t,x)} \int_t^T e^{-(\rho+\lambda)(s-t)}[U(c_s) + f_n(s, Y_s^{t,x})]ds. \tag{5.4}$$

By similar arguments as in (3.21)-(3.22), one can derive an uniform bound on $(\hat{v}_n, v_n)_n$:

$$\hat{v}_n(t,x,a) \;\leq\; K(e^{bt}x)^\gamma, \quad \forall (t,x,a) \in \mathcal{D}, \tag{5.5}$$

$$v_n(x) \;\leq\; Kx^\gamma, \quad \forall x \geq 0, \tag{5.6}$$

for some constant K independent of n. Moreover, under the condition (3.12), $\rho + \lambda > b\gamma$, therefore

$$e^{-(\rho+\lambda)(T-t)}\hat{v}_{n+1}(T, Y_T^{t,x}) \leq e^{-(\rho+\lambda)(T-t)}\hat{v}_{n+1}(T,x) \leq Kx^\gamma e^{(b\gamma-\rho-\gamma)T+(\rho+\gamma)t}$$

converges to zero exponentially fast and uniformly on x and t on compacts as $T \to \infty$. This shows that we can approximate $\hat{v}_{n+1}(t,x)$ by $\hat{v}_{n+1}^T(t,x)$ with any desired precision. On the other hand, (5.4) is a finite-horizon deterministic control problem well studied in the literature. The value function $\hat{v}_{n+1}^T(t,x)$ is the unique viscosity solution of the HJB equation

$$-(\rho+\lambda)\hat{v}_{n+1}^T + \frac{\partial \hat{v}_{n+1}^T}{\partial t} + \tilde{U}\left(\frac{\partial \hat{v}_{n+1}^T}{\partial x}\right) + f_n(t,x) = 0, \quad (t,x) \in [0,T] \times [\ell(a), \infty)$$

with boundary condition

$$\hat{v}_{n+1}^T(t,\ell(a)) = \int_t^T e^{-(\rho+\lambda)(s-t)} f_n(t,\ell(a))ds, \quad t \in [0,T]$$

and terminal condition $v_{n+1}^T(T,x) = 0$ for all $x \in [\ell(a),\infty)$. This equation can be approximated numerically by a standard backward discretization scheme as discussed for example in [2].

5.2 Convergence of the iterative decoupling algorithm

We now focus on the convergence of the sequence of functions $(\hat{v}_n, v_n)_n$ as n goes to infinity. Although we have an uniform bound (5.5) on $(\hat{v}_n, v_n)_n$, the equicontinuity of $(\hat{v}_n, v_n)_n$ seems much more difficult to establish in order to apply Ascoli-Arzela theorem and thus to get the convergence of the sequence $(\hat{v}_n, v_n)_n$. Instead, by means of dynamic programming arguments, we provide an autonomous probabilistic representation of $\hat{v}_n$ and v_n. Given $x \in \mathbb{R}_+$, we denote by $\mathcal{A}_n(x)$ the subset of controls $(\alpha, c) = ((\alpha_k)_k, (c_t)_t) \in \mathcal{A}(x)$ s.t. $\alpha_k = 0$ for $k \geq n+1$. In other words, $\mathcal{A}_n(x)$ is the set of admissible controls with at most n trading interventions and we have

$$\mathcal{A}_n(x) \;\subset\; \mathcal{A}_{n+1}(x) \;\subset\; \mathcal{A}(x).$$

We then have the following representation of v_n :

Proposition 5.1. For all $n \geq 0$, we have

$$v_n(x) \;=\; \sup_{(\alpha,c)\in\mathcal{A}_n(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t)dt\right], \quad x \geq 0, \tag{5.7}$$

Proof. We set for all $n \geq 0$,

$$w_n(x) \;=\; \sup_{(\alpha,c)\in\mathcal{A}_n(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t)dt\right], \quad x \geq 0. \tag{5.8}$$

A straightforward modification of the proof of the dynamic programming principle in Theorem A.2 enables us to show that under the hypotheses of this theorem (see its second part),

$$w_{n+1}(x) \;=\; \sup_{(a,c)\in\mathcal{A}_d(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)dt + e^{-\rho\tau_1} w_n(X_1^x)\right], \quad x \geq 0. \tag{5.9}$$

Then, by same arguments as in the derivation of relations (3.6), (3.9), the sequence of functions $(w_n)_n$ is given in inductive form by :

$$\begin{aligned} w_{n+1} &= \mathcal{H}\hat{w}_{n+1}, \quad \forall n \geq 0, \\ \hat{w}_{n+1}(t,x,a) &= \sup_{c\in\mathcal{C}_a(t,x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int w_n(Y_s^{t,x} + az)p(s,dz)\right] ds. \end{aligned}$$

From the definition of $(\hat{v}_n, v_n)$ and by induction starting from $w_0 = v_0$, we deduce that $\hat{w}_n = \hat{v}_n$, $w_n = v_n$, for all $n \geq 1$. This ends the proof. □

As a consequence, we can prove the convergence of the sequence of value functions.

Theorem 5.1. Under **(H1)-(H2)-(H3)**, (2.4), and (3.12), the sequence of functions $(\hat{v}_n, v_n)_{n\geq 0}$ converge uniformly on any compact subset of $\mathcal{D}$ and $\mathbb{R}_+$ to $(\hat{v}, v)$. More precisely, for any compact subset F and G of $\mathcal{D}$ and $\mathbb{R}_+$, there exist some positive constants C_F and C_G s.t.

$$0 \leq \sup_F(\hat{v} - \hat{v}_n) \leq C_F\, \delta^n, \tag{5.10}$$

$$0 \leq \sup_G(v - v_n) \leq C_G\, \delta^n, \tag{5.11}$$

where δ is defined in (3.14).

Proof. 1) From the dynamic programming principle (A.4) (see Remark A.1), for all $\varepsilon > 0$, $n \geq 1$, $x \in \mathbb{R}_+$, one can find $(\hat{\alpha}, \hat{c}) \in \mathcal{A}(x)$ s.t.

$$v(x) - \varepsilon \leq \mathbb{E}\left[\int_0^{\tau_n} e^{-\rho t} U(\hat{c}_t)dt + e^{-\rho\tau_n} v(X_n^x)\right]. \tag{5.12}$$

Now, observe that the "truncated" control $(\alpha^{(n)}, c^{(n)})$ defined by $\alpha_k^{(n)} = \hat{\alpha}_k 1_{k\leq n}$, $k \in \mathbb{N}^*$, $c_t^{(n)} = \hat{c}_t 1_{t\leq\tau_n}$, $t \geq 0$, lies in $\mathcal{A}_n(x)$. Then, from the representation (5.7) of v_n, we have

$$v_n(x) \geq \mathbb{E}\left[\int_0^{\tau_n} e^{-\rho t} U(\hat{c}_t)dt\right]. \tag{5.13}$$

Moreover, from (3.13) and (3.22), we have

$$\mathbb{E}\left[e^{-\rho\tau_n} v(X_n^x)\right] \leq K\mathbb{E}\left[e^{-\rho\tau_n}(X_n^x)^\gamma\right] \leq Kx^\gamma\delta^n. \tag{5.14}$$

Therefore, by noting also from (5.7) that $(v_n)_n$ is nondecreasing with $v_n \leq v$, and plugging (5.13)-(5.14) into (5.12), we obtain :

$$v(x) - \varepsilon - Kx^\gamma\delta^n \leq v_n(x) \leq v(x). \tag{5.15}$$

This proves the uniform convergence of v_n to v on any compact subset of $\mathbb{R}_+$, and the estimation (5.11).

2) From the definition of $\hat{v}_n$ and since v_n is a nondecreasing sequence converging to v, we clearly have $\hat{v}_n \leq \hat{v}_{n+1} \leq \hat{v}$. On the other hand, by definition of $\hat{v}$, for all $\varepsilon > 0$, $(t, x, a) \in \mathcal{D}$, one can find $c \in \mathcal{C}_a(t, x)$ s.t.

$$\hat{v}(t, x, a) - \varepsilon \leq \int_t^\infty e^{-(\rho+\lambda)(s-t)}\left[U(c_s) + \lambda\int v(Y_s^{t,x} + az)p(s, dz)\right]ds.$$

By using (5.15) and observing also that $Y_s^{t,x} \leq x$, we get :

$$\begin{aligned}
\hat{v}(t, x, a) - \varepsilon &\leq \int_t^\infty e^{-(\rho+\lambda)(s-t)}\left[U(c_s) + \lambda\int v_n(Y_s^{t,x} + az)p(s, dz)\right]ds \\
&\quad + \int_t^\infty e^{-(\rho+\lambda)(s-t)}\lambda\int\left[\varepsilon + K\delta^n(x + az)^\gamma\right]p(s, dz)ds \\
&\leq \hat{v}_{n+1}(x) + \frac{\varepsilon\lambda}{\rho+\lambda} + \lambda K\delta^n\int_t^\infty e^{-(\rho+\lambda)(s-t)}x^\gamma\frac{\kappa^\gamma}{\underline{z}^\gamma}e^{b\gamma s} \\
&\leq \hat{v}_{n+1}(x) + \frac{\varepsilon\lambda}{\rho+\lambda} + \lambda K\delta^n x^\gamma\frac{1}{\rho+\lambda-b\gamma}\frac{\kappa^\gamma e^{b\gamma t}}{\underline{z}^\gamma},
\end{aligned}$$

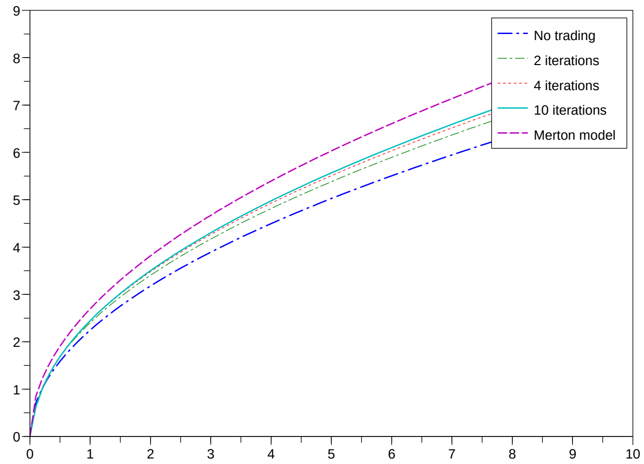

Figure 1: Convergence of the iterative algorithm for computing the value function in an illiquid market. The limiting value is smaller than that of the classical Merton problem due to the cost of liquidity.

where we used again (3.25). This proves the uniform convergence of $\hat{v}_n$ to $\hat{v}$ on any compact subset of $\mathcal{D}$, and the estimation (5.10). $\square$

5.3 Numerical illustration

Figure 1 shows the form of the value function and figure 2 that of the optimal investment policy in the partially observed Black-Scholes model (cf. example 2.1) with parameters $b = 0.1$ and $\sigma = 0.2$, obtained at different iterations of the numerical decoupling algorithm described earlier in this section. The utility function is $U(x) = \frac{x^\gamma}{\gamma}$ with $\gamma = 0.5$ and the other parameter values are $\rho = 0.4$, $\lambda = 1$ and $r = 0$ (no interest rates). The limiting value function lies between the solution corresponding to the model without trading $v(x) = K_0 x^\gamma$ and the value function of the Merton portfolio problem $v(x) = K_M x^\gamma$, where

$$K_0 = \frac{1}{\gamma}\left(\frac{1-\gamma}{\rho}\right)^{1-\gamma} \quad \text{and} \quad K_M = \frac{1}{\gamma}\left(\frac{1-\gamma}{\rho-\eta}\right)^{1-\gamma}, \quad \text{with} \quad \eta = \frac{b^2\gamma}{2\sigma^2(1-\gamma)}.$$

We observe the same qualitative behavior of the results as in Merton's model (the value function ressembles a power law and the optimal investment is a fraction of the total wealth), however due to the cost of liquidity the value function in an illiquid market is smaller than that of the Merton portfolio problem. The optimal proportion to invest in the risky asset is also very different: whereas in Merton's model the optimal fraction is equal to $\frac{b-r}{(1-\gamma)\sigma^2} = 5$, that is, the investor must borrow money to place more in the risky asset, in the illiquid market, as seen from figure 2, the optimal fraction is only about 0.15. This is due to the fact that the illiquid market is much more risky than the one where continuous trading in allowed because by the time the investor will have the next occasion to sell the asset its price may fall by an unpredictable amount. In particular, in the model of example 2.1, $\underline{z} = 1$ and therefore to ensure that the total wealth is always positive, the fraction to invest in the risky asset must not be greater than 1 (cf. equation (2.7)).

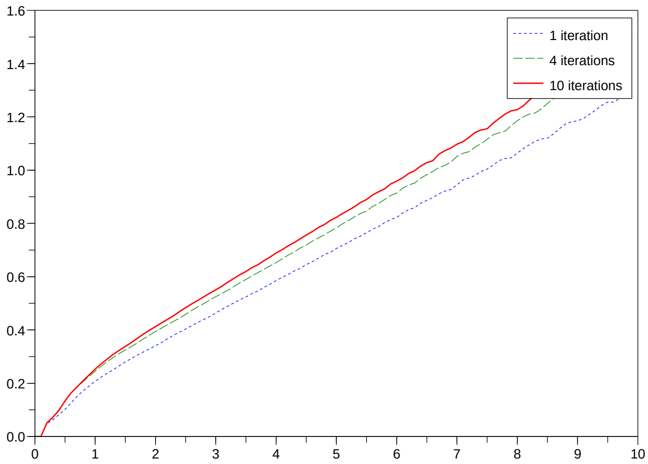

Figure 2: Convergence of the iterative algorithm for computing the optimal investment policy (the amount to invest in stock as a function of the total wealth at the trading date).

Appendix : Dynamic programming principle

In this section, we derive the dynamic programming principle for the *weak* formulation of the stochastic control problem (2.3) where one varies the probability spaces as well as controls.

Definition A.1. The space $\mathcal{U}^w$ of controls is the set of all 7-uples

$$(\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k\geq 1}, (Z_k)_{k\geq 1}, (\alpha_k)_{k\geq 1}, (c_t)_{t\geq 0})$$

satisfying the following:

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space.

(ii) $(\tau_k)_{k\geq 1}$ and $(Z_k)_{k\geq 1}$ satisfy the hypotheses **(H1)** and **(H2)** under $\mathbb{P}$. Let $\mathcal{G}_t$ denote the filtration of the marked point process $(\tau_k, Z_k)_{k\geq 1}$. This means in particular that $\mathcal{G}_{\tau_n} = \sigma\{(\tau_k, Z_k) : k \leq n\}$, for all $n \geq 1$ (cf. Theorem T30 in Appendix A2 in Brémaud [3]). By convention, $\tau_0 = 0$.

(iii) For each k, (α_k) is $\mathcal{G}_{\tau_{k-1}}$–measurable.

(iv) $(c_t)_{t\geq 0}$ is a nonnegative $\mathcal{G}_t$-predictable process.

The admissible consumption processes are characterized by the following result from [3, Theorem T34 in Appendix A2].

Lemma A.1. A process $(c_t)_{t\geq 0}$ is $\mathcal{G}_t$-predictable if and only if it admits the representation

$$c_t = \sum_{n\geq 0} C_n(t, \omega) 1_{\tau_n < t \leq \tau_{n+1}},$$

where, for every $n \geq 0$, the mapping $(t, \omega) \to C_n(t, \omega)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}_{\tau_n}$–measurable.

Let x be a.s. deterministic under $\mathbb{P}$. We denote by $\mathcal{A}^w(x)$ the set of all x-admissible controls: the subset of $\mathcal{U}^w$ containing all controls for which $\mathbb{P}[X_k^x \geq 0, \forall k \geq 1] = 1$, where X_k^x is defined by (2.1). $\mathcal{A}^w(x)$ is clearly non-empty for all $x \geq 0$.

The value function of the stochastic control problem (2.3) is now defined by

$$v(x) = \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right] \tag{A.1}$$

Theorem A.2 (Dynamic programming principle)**.** The value function defined in (A.1) satisfies

$$v(x) \leq \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(X_1^x)\right], \quad x \geq 0. \tag{A.2}$$

If, in addition, the hypotheses **(H3)**, (2.4) and (3.12) are satisfied then

$$v(x) = \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(X_1^x)\right], \quad x \geq 0. \tag{A.3}$$

Proof. 1. First part. Since $\mathbb{P}[Z_1 < 0] > 0$, the only admissible policy for $x = 0$ is $c_t \equiv 0$ and $\alpha_k \equiv 0$. Therefore, $v(0) = 0$ and (A.2) is trivially satisfied for $x = 0$. On the other hand, since by Lemma 4.1, v is nondecreasing and concave on $\mathbb{R}_+$, either $v(x) < \infty$ for all $x > 0$ or $v(x) = \infty$ for all $x > 0$, and in the latter case (A.2) is once again trivially satisfied (take the control $c_t \equiv 0$ and $\alpha_k \equiv 0$). Therefore, in this proof we suppose w.l.o.g. that $v(x) < \infty$, all $x \geq 0$.

Denote the right-hand side of (A.2) by $V(x)$. In this part we want to show that $v(x) \leq V(x)$, all $x \geq 0$.

Let $\varepsilon > 0$, $x \geq 0$. There is an element

$$u := (\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k\geq 1}, (Z_k)_{k\geq 1}, (\alpha_k)_{k\geq 1}, (c_t)_{t\geq 0}) \in \mathcal{A}^w(x),$$

such that

$$\begin{aligned} v(x) - \varepsilon &\leq \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right] \\ &= \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt\right] + \mathbb{E}\left\{e^{-\rho \tau_1} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_{\tau_1 + t}) dt \middle| \mathcal{G}_{\tau_1}\right]\right\} \end{aligned}$$

Let $\tilde{\tau}_k = \tau_{k+1} - \tau_1$, $\tilde{Z}_k = Z_{k+1}$, $\tilde{\alpha}_k = \alpha_{k+1}$ and $\tilde{c}_t = c_{\tau_1 + t}$. If we are able to show that $X_1^x = x - \int_0^{\tau_1} c_t dt + \alpha_1 Z_1$ is a.s. deterministic under $\mathbb{P}(\cdot|\mathcal{G}_{\tau_1})$ and that

$$\tilde{u} := (\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{G}_{\tau_1}), (\tilde{\tau}_k)_{k\geq 1}, (\tilde{Z}_k)_{k\geq 1}, (\tilde{\alpha}_k)_{k\geq 1}, (\tilde{c}_t)_{t\geq 0}) \in \mathcal{A}^w(X_1^x),$$

it will follow that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_{\tau_1 + t}) dt \middle| \mathcal{G}_{\tau_1}\right] \leq v(X_1^x), \quad \mathbb{P}(\cdot|\mathcal{G}_{\tau_1}) - a.s.,$$

and therefore $v(x) \leq V(x)$.

By Lemma A.1,

$$X_1^x = x - \int_0^{\tau_1} C_0(t) dt + \alpha_1 Z_1$$

for some measurable deterministic function C_0. Therefore, X_1^x is a.s. deterministic under $\mathbb{P}(\cdot|\mathcal{G}_{\tau_1})$. Conditions (i) and (ii) of Definition A.1 are clearly satisfied. Since Z_1 and τ_1 are almost surely deterministic under $\mathbb{P}(\cdot|\mathcal{G}_{\tau_1})$, $\tilde{\alpha}_n$ is measurable with respect to $\sigma\{(\tau_k, Z_k) : 2 \leq k \leq n+1\}$, and so with respect to $\mathcal{G}_{\tau_{n+1}}$, which proves condition (iii). To prove condition (iv), fix some $n \geq 0$. By Lemma A.1,

$$\tilde{c}_t 1_{\tilde{\tau}_n < t \leq \tilde{\tau}_{n+1}} = c_{t+\tau_1} 1_{\tau_{n+1} < t+\tau_1 \leq \tilde{\tau}_{n+2}} \qquad = C^{n+1}(t+\tau_1, \omega) 1_{\tau_{n+1} < t+\tau_1 \leq \tilde{\tau}_{n+2}},$$

where C^{n+1} is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}_{\tau_{n+1}}$–measurable. Therefore (cf. Theorem 1.7 in [15]), there exists a measurable mapping $f^{n+1} : \mathbb{R}^{2n+3} \to \mathbb{R}_+$ such that

$$\tilde{c}_t 1_{\tilde{\tau}_n < t \leq \tilde{\tau}_{n+1}} = f^{n+1}(t+\tau_1, \tau_1, Z_1, \ldots, \tau_{n+1}, Z_{n+1}) 1_{\tau_{n+1} < t+\tau_1 \leq \tilde{\tau}_{n+2}}.$$

Since Z_1 and τ_1 are $\mathbb{P}(\cdot|\mathcal{G}_{\tau_1})$–a.s. deterministic, $f^{n+1}(t+\tau_1, \tau_1, Z_1, \ldots, \tau_{n+1}, Z_{n+1})$ is $\mathcal{B}(\mathbb{R}_+) \otimes \tilde{\mathcal{G}}_{\tilde{\tau}_n}$-measurable and we conclude, once again by Lemma A.1, that condition (iv) of Definition A.1 is satisfied and $\tilde{u} \in \mathcal{U}^w$. Finally, from the admissibility of $u \in \mathcal{A}^w(x)$, it is straightforward to check that $\tilde{u} \in \mathcal{A}(X_1^x)$.

2. Second part. Let us now prove that $v(x) \geq V(x)$, all $x \geq 0$ under **(H3)**, (2.4) and (3.12). First, we notice under these conditions, and by the arguments of Corollary 3.1, that $V(x) \leq Kx^\gamma$, for all $x \geq 0$, and in particular is finite. Hence, for all $x \geq 0$, $\varepsilon > 0$, one may find

$$u = (\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k\geq1}, (Z_k)_{k\geq1}, (\alpha_k)_{k\geq1}, (c_t)_{t\geq0}) \in \mathcal{A}^w(x)$$

such that

$$V(x) \leq \frac{\varepsilon}{3} + \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) + e^{-\rho\tau_1} v(X_1^x)\right]$$

with $X_1^x = x + \alpha_1 Z_1 - \int_0^{\tau_1} c_t dt$.

Since the value function is nondecreasing and continuous on $[0, \infty)$, one can choose a sequence of measurable sets $\{B_j\}_{j\geq1}$ such that $\bigcup_{j\geq1} B_j = \mathbb{R}_+$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and whenever $x, y \in B_j$, $|v(x) - v(y)| \leq \frac{\varepsilon}{3}$. For every j, put $x_j = \inf B_j$ and a choose a control

$$u_j = (\Omega_j, \mathcal{F}_j, \mathbb{P}_j, (\tau_k^j)_{k\geq1}, (Z_k^j)_{k\geq1}, (\alpha_k^j)_{k\geq1}, (c_t^j)_{t\geq0}) \in \mathcal{A}^w(x_j)$$

such that

$$v(x_j) \leq \frac{\varepsilon}{3} + \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t^j) dt\right].$$

Note that $u_j \in \mathcal{A}^w(x')$ for every $x' \in B_j$.

By the same argument as in the proof of part 1, for every j, one can find a sequence $(f_n^j)_{n\geq0}$, $f_n^j : \mathbb{R}^{2n+1} \to \mathbb{R}_+$ measurable such that

$$c_t^j = \sum_{n\geq0} f_n^j(t, \tau_1^j, Z_1^j, \ldots, \tau_n^j, Z_n^j) 1_{\tau_n^j < t \leq \tau_{n+1}^j}$$

and a sequence $(g_n^j)_{n\geq1}$, $g_n^j : \mathbb{R}^{2n-1} \to \mathbb{R}$ measurable such that

$$\alpha_n^j = g_n^j(\tau_1^j, Z_1^j, \ldots, \tau_{n-1}^j, Z_{n-1}^j).$$

Now define the new control $\tilde{u}$ via

$$\tilde{u} = (\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k\geq1}, (Z_k)_{k\geq1}, (\tilde{\alpha}_k)_{k\geq1}, (\tilde{c}_t)_{t\geq0}),$$

where

$$
\begin{aligned}
\tilde{\alpha}_1 &= \alpha_1, \\
\tilde{\alpha}_n &= \sum_j 1_{X_1^x \in B_j} g_{n-1}^j(\tau_2, Z_2, \ldots, \tau_{n-1}, Z_{n-1}), \quad n \geq 2, \\
\tilde{c}_t &= c_t 1_{t \leq \tau_1} + \sum_j 1_{X_1^x \in B_j} \sum_{n \geq 0} f_n^j(t-\tau_1, \tau_2 - \tau_1, Z_2, \ldots, \tau_{n+1} - \tau_1, Z_{n+1}) 1_{\tau_{n+1} < t \leq \tau_{n+2}}.
\end{aligned}
$$

By construction, $\tilde{u} \in \mathcal{A}^w(x)$. Finally,

$$
\begin{aligned}
v(x) &\geq \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(\tilde{c}_t) dt\right] \\
&= \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(\tilde{c}_t) dt + e^{-\rho \tau_1} \mathbb{E}\left\{\int_0^\infty e^{-\rho t} U(\tilde{c}_{\tau_1 + t}) dt \middle| \mathcal{G}_{\tau_1}\right\}\right] \\
&\geq \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(\tilde{c}_t) dt + e^{-\rho \tau_1} \sum_j v(x_j) 1_{X_1^x \in B_j}\right] - \frac{\varepsilon}{3} \\
&\geq \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(\tilde{c}_t) dt + e^{-\rho \tau_1} v(X_1^x)\right] - \frac{2\varepsilon}{3} \\
&\geq V(x) - \varepsilon.
\end{aligned}
$$

Since the choice of $\varepsilon > 0$ was arbitrary, the proof is complete. □

Remark A.1. A straightforward modification of the above proof allows to establish the following modified version of the dynamic programming principle : for every $n \geq 1$,

$$
v(x) = \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^{\tau_n} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_n} v(X_n^x)\right], \quad x \geq 0. \tag{A.4}
$$

Remark A.2. Finally, we note that the dynamic programming principles (A.3) can be formulated on a single probability space. Indeed, from lemma A.1 and the mesurability condition on (α_k), for every admissible control

$$
u := (\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k \geq 1}, (Z_k)_{k \geq 1}, (\alpha_k)_{k \geq 1}, (c_t)_{t \geq 0}),
$$

α_1 is a deterministic constant and $c_t = \tilde{c}(t) 1_{t < \tau_1}$ for some deterministic function $\tilde{c}(t)$. Therefore, we can fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the conditions (i) and (ii) of definition A.1 and equation (A.3) will take the form

$$
v(x) = \sup_{A_d(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(X_1^x)\right], \quad x \geq 0, \tag{A.5}
$$

where $A_d(x)$ is the set of deterministic controls defined in equation (3.2).

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